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A THEORETICAL STUDY OF UNSTEADY DROPLET BURNING: TRANSIENTS AND PERIODIC SOLUTIONS

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Prepared by: Warren C. Strahle

Warren C. Strahle

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Approved by: Luigi Crocco

Luigi Crocco
Robert H. Goddard Professor
of Jet Propulsion

and David T. Harrje

David T. Harrje
Research Aeronautical Engineer

Guggenheim Laboratories for the Aerospace Propulsion Sciences
Department of Aeronautical Engineering
PRINCETON UNIVERSITY, N.J.

1885953

ABSTRACT

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The burning of liquid droplets in a high performance combustor is essentially an unsteady process. However, recognition is given to the fact that unsteady analysis has rarely been given to the problem of rapid burning and vaporization.

Through unsteady analysis conditions are established whereby it is possible to consider that droplets do vaporize and burn in a nearly steady state during most of their lifetime. Periodic solutions are then obtained when the droplet is burning in an unsteady acoustic field of the ambient gas.

It is shown from the periodic solutions that extremely strong response of the vaporization rate and burning rate can occur throughout the frequency spectrum. Application to the problem of combustion instability is indicated.

Author

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NOMENCLATURE

a	nose radius of liquid body
A	$(1/B_\infty) - \tau$ or, when subscripted, constant of integration in high frequency analyses
B	Spalding transfer number, $c_p^* (T_f^* - \tau^*)/\Delta\ell^*$
B_∞	$c_p^* T_\infty^* / \Delta\ell^*$
$B(T)$	pre-exponential frequency factor in chemical kinetics
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
c_L	specific heat of liquid
C	$1 - (r_L/r_f)$
D_{12}	binary diffusion coefficient
E	activation energy of reaction kinetics
F	boundary layer stream function variable
g	low frequency expansion functions of P
h	enthalpy or low frequency expansion coefficients of σ
H, S, V_K, R, U, W_K	high frequency expansion functions for P, σ, Y_K
h, s, v_K, r, u, w_K	high frequency expansion functions for H, S, V_K, R, U, W_K
i	complex variable, $\sqrt{-1}$
j	stoichiometric mass ratio, oxidizer to fuel
$\hat{j}$	index for axisymmetric or two-dimensional flow
k_K	low frequency expansion coefficients of Y_K
K	ratio of two characteristic times or λ^*/π^*
$\Delta\ell$	latent heat of vaporization

Le	Lewis number
$\mathcal{L}$	linear operator or perturbation in $1/Le$
m	mass flow rate per unit area
$\hat{m}$	total mass flow rate $/4\pi$
M	Mach number or space dependent part of m perturbation
n	intrinsic coordinate normal to body unless otherwise noted
$O[\ ]$	the order of $[\ ]$
p	pressure
P	space dependent part of ψ perturbation
Pr	Prandtl number
q	stoichiometric heat of reaction
q_{CH}	heat release per unit mass of fluid due to chemical reaction
Q_i	i th component of heat flux vector
r	radial variable
R_K	gas constant for specie K
R_c	perturbation amplitude of r_c
Re	Reynolds number
s	intrinsic coordinate running along body from the stagnation point
Sc	Schmidt number
t	time
T	temperature
u	velocity in s direction
U	velocity in inviscid stream

v	velocity in n direction
$\bar{v}$	flat plate v-velocity variable
$\hat{v}$	stagnation point v-velocity variable
v_i	i^{th} component of mass weighted average gas velocity
v_{K_i}	i^{th} component of absolute velocity of specie K
V_{K_i}	i^{th} component of diffusion velocity of specie K
W	molecular weight
x_i	i^{th} space variable
$\breve{x}$	variable along axis of liquid body
y	boundary layer variable in n direction
Y_K	mass fraction of K^{th} specie
y_K	space dependent part of Y_K perturbation
z	stagnation point boundary layer variable
α	high frequency variable
β_1, β_2	high frequency boundary layer variables
γ	ratio of specific heats, c_p^*/c_v^*
ϵ	small perturbation parameter
ζ	stagnation point boundary layer variable
η	boundary layer variable
θ	angle between $\vec{s}$ and body axis
κ	body curvature or perturbation in thermal conductivity
λ	thermal conductivity
μ, μ_2	viscosity and second visocity coefficients

$\hat{\mu}$	boundary layer variable
ξ	low frequency boundary layer variable or space dependent part of ρ perturbation
ρ	density
σ	space dependent part of T perturbation
τ	liquid temperature or time
τ_{ij}	shear stress tensor
φ	space dependent part of p perturbation
Φ	dissipation function
Ψ	stream function
ω	frequency
$\hat{\omega}$	temperature exponent in transport property laws (subscripted by appropriate transport property)
w_K	rate of mass generation of specie K per unit mass of fluid

Superscripts

$\rightarrow$	vectorial quantity
$'$	total differentiation with respect to independent variable
$*$	dimensional quantity
(s)	steady state solution
$-$	steady state quantity unless otherwise noted
$\smile$	perturbation quantity, time and space dependent
(i)	i^{th} term of expansion in powers of a parameter or variable.

Subscripts

c	outer boundary where conditions are specified
CH	chemical
d	droplet
dif	diffusion
f	flame
F	fuel
g	gas
HU	heat-up
i	"inner"
IC	initial conditions
K	specie K
L	liquid
L/g	ratio of quantities, liquid to gas
O	oxidizer
o	initial conditions or "outer"
r or i	real or imaginary quantities
r	reference quantities
W	wall or liquid surface
WB	wet bulb
F_i	fuel side of flame, at wall
F_o	fuel side of flame, at flame
O_i	oxidizer side of flame, at flame
∞	ambient conditions or at infinity of boundary layer flow
(i)	i^{th} term in expansion of powers of δM

Other quantities, superscripts, or subscripts are defined at their origin in the text and appendices.

INTRODUCTION

Many important combustion and propulsion systems involve the injection of a liquid fuel and/or oxidizer into the combustion chamber. Subsequent to the injection process atomization, vaporization, ignition, and burning of the substance occurs. In view of the high temperatures usually found in combustion chambers this conversion process from liquid to burned gas takes place extremely rapidly, and from the frame of reference of a liquid droplet this conversion process is essentially unsteady. The droplet, injected under high pressure and low temperature, will heat up in time. In view of the vaporization of the liquid, the liquid-gas interface must contract in time. The conditions of the ambient gas immediately after droplet formation do not correspond to the distributions of temperature, velocity, etc. which exist around the droplet after sufficient time has elapsed for the initial distribution to "relax". Secondary atomization may occur if the gas flow field in which the droplet exists creates dynamic forces strong enough to overcome the cohesive surface tension forces. Combustors are usually quite turbulent and the droplet is therefore burning in a randomly fluctuating field. The ambient conditions in the gas vary as the droplet moves through the combustor. Finally, burning may take place in the presence of organized acoustic oscillations of the chamber gas, and oscillatory combustion has long been a mystery in reference to unstable combustion systems.

It has been the practice, however, to always treat the droplet burning process as essentially quasi-steady. That is, at any instant of time the burning rate and heat-up rate are calculated from equations based on the instantaneous average conditions in the surroundings which are assumed stationary in time. The liquid-gas interface is assumed stationary. From the theoretical standpoint this always amounts to neglect of time rates of storage of mass, energy, and momentum in the diffusion field surrounding a droplet. The reason that this assumption is made is that the time dependent problem is essentially non-linear and extremely difficult. Solutions to problems which even separate the above types of unsteadiness have never been obtained for high temperature, rapid burning or vaporization.

A great deal of work, both theoretical and experimental, has been performed through the quasi-steady approach; excellent reviews have been published (1, 2)[1]. No attempt will be made to give such a review here.

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1. Numbers in parentheses refer to References on page 154.

However, some essentials of these theories will be derived as a natural course toward the unsteady problem.

The purposes here are to first investigate errors introduced in some of the quasi-steady assumptions when an isolated droplet is burning in a non-oscillatory ambient gas. The second is to gain some insight into oscillatory combustion by obtaining a periodic solution to the burning problem when a periodic sound wave is propagating through the ambient gas. Much more attention will be paid to the second purpose than the first. It will be seen that there is great difficulty in obtaining a mathematically well-posed problem in the unsteady state unless a rather complex theory is built for the steady state. These difficulties always arise when some of the essential physics of the problem are removed for treatment of the steady state problem and then extended to the unsteady state.

The overall questions which are examined are 1.) under what conditions in a non-oscillatory field will a droplet burn during most of it's lifetime in essentially a quasi-steady fashion? and 2.) given the first set of conditions is it possible that the periodic burning and/or vaporization rate under acoustic periodic oscillations of the ambient gas will tend to amplify or damp the acoustic oscillations?

CHAPTER I

FUNDAMENTALS AND THE STAGNANT-FILM QUASI-STEADY THEORY

A. Basic Equations

In order that all the subsequent problems to be treated will have some measure of apparent order it is best that the full set of differential equations describing the phenomena be set down at the beginning. The system for consideration is a liquid body at rest immersed in a flowing or stagnant gas. The liquid is volatile and the vapor is capable of chemical reactions with the ambient gas. The following initial assumptions will be made concerning the gas:

1. It is a continuum.
2. There is heat transfer by conduction only - no radiation is considered.
3. There are no overall mass sources.[1]
4. Mass diffusion occurs by a concentration gradient only. Thermal and pressure diffusion are neglected. The validity of this will rest on further assumptions to be made.

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1. However, there may be individual species mass sources due to the presence of combustion.

5. The mixture at any point in the gas is a binary fluid. Only two species are allowed, e.g., fuel vapor and a ficticious single component product gas. Thus, the extreme complexity of multicomponent diffusion is avoided.
6. There are no overall or preferential species body forces. (gravitational, centrifugal, electromagnetic, etc.).
7. The flow is locally laminar.

Then following Tsien (3) and Penner (4) the governing equations are given in Cartesian tensor notation

Continuity

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x_i^*}\left(\rho^* v_i^*\right) = 0 \tag{1.1}$$

Species Continuity $\quad (K = 1, 2)$

$$\frac{D Y_K}{D t^*} + \frac{1}{\rho^*}\frac{\partial}{\partial x_i^*}\left(\rho^* Y_K V_{K_i}^*\right) = w_K^* \tag{1.2}$$

Momentum

$$\frac{D v_j^*}{D t^*} = \left(\frac{\partial}{\partial t^*} + v_i^* \frac{\partial}{\partial x_i^*}\right) v_j^* = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial x_j^*} + \frac{1}{\rho^*}\frac{\partial \tau_{ji}^*}{\partial x_i^*} \tag{1.3}$$

Energy

$$\frac{D h^*}{D t^*} = q_{CH}^* + \frac{1}{\rho^*}\frac{\partial Q_i^*}{\partial x_i^*} + \frac{1}{\rho^*}\left(\frac{Dp^*}{Dt^*} + \Phi^*\right) \tag{1.4}$$

The diffusion velocity, $V_{K_i}^*$, is given by

$$V_{K_i}^* = -D_{12}^* \frac{\partial}{\partial x_i^*}\left(\ln Y_K\right) \tag{1.5}$$

and the relation of the mass weighted average velocity, v_i^*[1],

1. This is the velocity of the center of mass of a fluid element.

and the absolute velocity of species K, $V_{K_i}^*$, is

$$V_{K_i}^* = v_i^* + V_{K_i}^* \tag{1.6}$$

The shear stress tensor, τ_{ij}^*, is given by

$$\tau_{ij}^* = \mu^* \left(\frac{\partial v_i^*}{\partial x_j^*} + \frac{\partial v_j^*}{\partial x_i^*} \right) + \mu_2^* \delta_{ij} \frac{\partial v_l^*}{\partial x_l^*} \tag{1.7}$$

$$\delta_{ij} = \begin{matrix} 1 & j = i \\ 0 & j \neq i \end{matrix}$$

The heat flux vector, Q_i^*, contains both heat conduction and diffusion and is given by

$$Q_i^* = \lambda^* \frac{\partial T^*}{\partial x_i^*} - \rho^* \sum_{K=1}^{2} Y_K h_K^* V_{K_i}^* \tag{1.8}$$

The dissipation function, Φ^*, is given by

$$\Phi^* = \frac{\mu^*}{2} \left(\frac{\partial v_i^*}{\partial x_j^*} + \frac{\partial v_j^*}{\partial x_i^*} \right)^2 + \mu_2 \left(\frac{\partial v_i^*}{\partial x_i^*} \right)^2 \tag{1.9}$$

The energy equation, Eq. (1.4), refers only to sensible enthalpy and q_{CH}^* must therefore be included. w_K^* and q_{CH}^* can only be determined if chemical kinetics are considered.

Except where noted the following additional assumptions will hold through most of the work:

8. Each constituent of the mixture will obey the perfect gas law, each component being thermally and calorically perfect.
9. The viscosities, $\rho^* D_{12}^*$ product, and thermal conductivity will be specified functions of temperature only.

10. The specific heats of each species will be the same, implying equal molecular weights and making assumption 4 above plausible.

Then the state equation is

$$
\begin{aligned}
p_K^* &= R_K^* \rho_K^* T = R^* \rho_K^* T \\
p^* &= p_1^* + p_2^* \\
\rho^* &= \rho_1^* + \rho_2^* \\
h_K^* &= c_{p_K}^* T^* = c_p^* T^* = h^* \\
\rho_K^* &= \rho^* Y_K
\end{aligned}
\tag{1.10}
$$

The transport property laws are

$$
\begin{aligned}
&\mu^* \propto T^{*\hat{\omega}_\mu} \qquad \mu_2 \propto T^{*\hat{\omega}_{\mu_2}} \\
&\rho^* D_{12}^* \propto T^{*\hat{\omega}_{D_{12}}} \qquad \lambda^* \propto T^{*\hat{\omega}_\lambda}
\end{aligned}
\tag{1.11}
$$

Eqs. (1.1)-(1.11) together with appropriate boundary conditions are sufficient to specify the gas dynamic problem if chemical kinetics expressions are used to determine w_K^*. In what follows it will usually be possible to bypass the chemical kinetics problem. In fact, w_K^* and consequently q_{CH}^* will be set equal to zero for the present and consideration will be given only to regions where chemical reaction is absent. Regions where it does occur will be treated later.

Eqs. (1.10) and (1.6) have as a consequence that

$$\sum_{K=1}^{2} Y_K h_K^* V_{K_i}^* = 0$$

so that Eq. (1.8) is accordingly simplified.

Non-dimensionalization of the above equations is affected by introducing a reference time, velocity, density, temperature, length, and viscosity and by further assuming

$$\hat{\omega}_{\mu} = \hat{\omega}_{D_{12}} = \hat{\omega}_{\lambda} = \hat{\omega}_{\mu_2}$$

Then

$$t = t^*/t_r^* \qquad M = U_r^*/\sqrt{\gamma R^* T_r^*}$$

$$v_i = v_i^*/U_r^* \qquad \gamma = c_p^*/c_v^*$$

$$\rho = \rho^*/\rho_r^* \qquad Re = \frac{U_r^* \rho_r^* x_r^*}{\mu_r^*}$$

$$T = T^*/T_r^*$$

$$x_i = x_i^*/x_r^* \qquad Sc = \frac{\mu^*}{\rho^* D_{12}^*}$$

$$\mu = \mu^*/\mu_r^* \qquad Pr = \frac{\mu^* c_p^*}{\lambda^*}$$

$$K = t_r^* U_r^*/x_r^* \qquad Le = Sc/Pr$$

$$p = p^*/\rho_r^* U_r^{*2} \qquad m = \frac{\rho^* v^*}{\rho_r^* U_r^*}$$

$$\tau_{ij} = \tau_{ij}^*/\rho_r^* U_r^{*2}$$

These reference quantities are as yet undefined and will be chosen differently for different applications. Continuity becomes

$$\frac{1}{K}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \qquad (1.12)$$

Combination of Eq's. (1.2) and (1.5) yields

$$\frac{1}{K}\frac{\partial Y_K}{\partial t} + v_i \frac{\partial Y_K}{\partial x_i} = \frac{1}{\rho \,\mathrm{Re}\,\mathrm{Sc}} \frac{\partial}{\partial x_i}\left(\mu \frac{\partial Y_K}{\partial x_i}\right) \tag{1.13}$$

Combination of Eq's. (1.5) and (1.6) yields a mass transfer relation

$$\frac{\mu}{\mathrm{Re}\,\mathrm{Sc}} \frac{\partial Y_K}{\partial x_i} = \rho v_i Y_K - \rho_K v_{K_i} = m_i Y_K - m_{K_i} \tag{1.14}$$

Momentum, Eq. (1.3), becomes

$$\frac{1}{K}\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} \tag{1.15}$$

where

$$\tau_{ij} = \frac{\mu}{\mathrm{Re}}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{\mu_2}{\mathrm{Re}} \delta_{ij} \frac{\partial v_l}{\partial x_l} \tag{1.16}$$

Combination of Eq's. (1.8) and (1.10) yields the energy equation

$$\frac{1}{K}\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \frac{1}{\rho \,\mathrm{Re}\,\mathrm{Pr}} \frac{\partial}{\partial x_i}\left(\mu \frac{\partial T}{\partial x_i}\right) + \frac{\Phi}{\rho}$$
$$+ \frac{M_r^2}{\rho}(\gamma - 1)\left[\frac{1}{K}\frac{\partial p}{\partial t} + v_i \frac{\partial p}{\partial x_i}\right] \tag{1.17}$$

where the dissipation function, Φ, is

$$\Phi = \frac{M_r^2(\gamma-1)}{\mathrm{Re}}\left\{\frac{\mu}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 + \mu_2\left(\frac{\partial v_i}{\partial x_i}\right)^2\right\} \tag{1.18}$$

The state equation, Eq. (1.10), is

$$\gamma M_r^2 p = \rho T \qquad\qquad \mathcal{R} = T \tag{1.19}$$

which holds also for each specie. Finally, the transport laws are incorporated into the statement

$$\mu = T^{\hat{\omega}} \qquad \mu_2 = f(\mu) \tag{1.20}$$

It will be found that the restriction on μ_2 will never have to be used in what follows but is included here for completeness.

Now Eq's. (1.12)-(1.20) completely specify the gas phase problem except for specification of the reference quantities, $\hat{\omega}$, and the boundary conditions for each individual problem to be treated.

B. Quasi-Steady Spherically Symmetric Burning Theory

The most common approach to droplet burning theory is to assume spherical symmetry. The coordinate system is shown in Figure 1a. A spherical fuel droplet is assumed burning in a stagnant ambient gas which contains an oxidizer. Fuel vaporizes and transfers outward to meet the inward diffusing oxidizer. Reaction takes place at an infinitely thin spherical sheet, requiring infinitely fast reaction kinetics. The process is maintained by the heat release at the spherical sheet heat source (flame) causing a temperature in excess of the droplet and usually of the ambient temperatures. The outer ambient boundary conditions are usually considered to lie at infinity. On either side of the heat source there are only either oxidizer and inert or fuel vapor and inert. Therefore, the reaction must take place at the stoichiometric mass feeding rate and the mass fractions of fuel and oxidizer must be zero at the flame.

A brief development of the theory will be given since it is essential to what follows. The differential equations generally used to describe this process in the gas phase are the following:

Continuity

$$\frac{d}{dr}(\rho v r^2) = 0 \tag{1.21a}$$

Species Continuity

$$\rho v r^2 \frac{dY_K}{dr} = \frac{1}{Le} \frac{d}{dr}\left(r^2 \frac{dY_K}{dr}\right) \tag{1.21b}$$

Momentum

$$p = \text{constant} \tag{1.21c}$$

Energy

$$\rho v r^2 \frac{dT}{dr} = \frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) \tag{1.21d}$$

State

$$\gamma M_r^2 p = \rho T = 1 \tag{1.21e}$$

These equations can be obtained from Eq's. (1.12)-(1.20) by the following assumptions:

1. Spherical symmetry exists.
2. A steady state exists. t_r^* is taken as the droplet lifetime. u_r^* is taken as the diffusion velocity, $\mu_r^*/\rho_r^* x_r^*$, where is, say, the droplet radius. Then disregarding the initial ignition and heat-up processes the assumption is $K \gg 1$; the diffusion time is very much shorter than the droplet lifetime.
3. Gas speeds are very much less than the speed of sound. Take $u_r^* = \sqrt{R^* T_r^*}$ where T_r^* is a typical temperature of the system and u_r^* is then approximately a sound speed. Pressure then becomes automatically referred to a typical pressure in the system. Then from Eq's. (1.15) and (1.16) it can be seen that if the Reynolds number based on the speed of sound in sufficiently large, the pressure is essentially constant throughout the film.
4. $\hat{\omega} = 0$ and the reference transport properties are taken at some average film value.

The reference quantities in Eqs. (1.21a - c) are chosen to be

$$x_r^* = r_{L_0}^* \qquad u_r^* = \bar{\lambda}^* / \rho_\infty^* c_p^* r_{L_0}^*$$

$$\rho_r^* = \rho_\infty^* \qquad T_r^* = T_\infty^*$$

The boundary conditions to complete the problem specification are usually written under the following additional assumptions:

5. The flame zone is collapsed to an infinitely thin spherical sheet heat source.
6. The liquid is at a uniform temperature and in a steady state.
7. As a consequence of 6 all heat transferred to the droplet goes toward vaporization, not liquid heat-up.
8. The liquid-gas interface is in equilibrium. That is, the partial pressure of the fuel vapor immediately adjacent to the liquid corresponds to the equilibrium saturated vapor pressure of the fuel at the liquid temperature[1].
9. The ratio of liquid to gas density at the interface is very large so that the gas velocity is much greater than the liquid surface regression velocity.
10. Le = 1

Then in dimensionless form the boundary conditions at the surface become

Heat Transfer at Droplet Surface

$$\left.\frac{dT}{dr}\right)_{r_L} = \frac{\rho v)_{r_L}}{B_\infty} \qquad (1.22\ a)$$

1. This assumes infinitely fast evaporation kinetics.

Mass Transfer at Droplet Surface

$$\left.\frac{dY_F}{dr}\right)_{r_L} = \left[\rho v\,(Y_F - 1)\right]_{r_L} \qquad (1.22\ b)$$

Boundary Values

$$T(r_L) = \tau \qquad\qquad Y_F(r_L) = Y_{F_L} \qquad (1.22\ c)$$

At the flame

Stoichiometric Condition

$$\left.\frac{dY_F}{dr}\right)_{r_f} j + \left.\frac{dY_O}{dr}\right)_{r_f} = 0 \qquad (1.23\ a)$$

Heat Generation

$$\left.\frac{dT}{dr}\right)_{r_f\ \text{Fuel Side}} - \left.\frac{dT}{dr}\right)_{r_f\ \text{Ox. Side}} + q\left.\frac{dY_F}{dr}\right)_{r_f} = 0 \qquad (1.23\ b)$$

Boundary Values

$T(r_f) = T_f$ is continuous

$Y_F(r_f) = Y_O(r_f) = 0$

$\rho v)_{r_f}$ is continuous (1.23 c)

At infinity

$$T(\infty) = 1$$

$$Y_O(\infty) = Y_{O_\infty} \qquad (1.24)$$

Eq's. (1.22 b) and (1.23 a) are obtained by evaluating Eq. (1.14) at the surface and flame respectively and noting that at the surface the only species which flows is the fuel so that $V_F = V$ and by noting that at the flame $Y_F = Y_O = 0$. The heat of reaction in Eq. (1.23 b) is independent of T_f because of the equality of specific heats assumption and is therefore known for any particular fuel.

This formulation of the problem is well known although the general approach here is a bit different than published previously. For the steady state problem it is overrestricted in the number of assumptions actually needed to obtain a rather simple solution. For instance assumptions 2, 5, and 10 of Section A and assumptions 4, 5, 6, 7, 8, and 9 of this section have been individually or collectively relaxed in the literature. As previously mentioned, reviews of this literature have been published (1, 2). However, much of the essential physics for a great number of problems is contained in this formulation and in view of the physics left out (spherical symmetry cannot exist) it is considered that refinements are somewhat futile. In the unsteady state some of these assumptions become essential for analytical treatment, but again it will be found that some may be relaxed.

Eqs. (1.21) subject to Eqs (1.22)-(1.24) may be integrated to yield the following well known results:

$$T_f = \frac{q \dfrac{Y_{O\infty}}{j} + \tau \dfrac{Y_{O\infty}}{j Y_{F_L}} + 1}{1 + \dfrac{Y_{O\infty}}{j Y_{F_L}}} \qquad (1.25\ a)$$

$$Y_{F_L} = B/(1+B) \tag{1.25 b}$$

$$\hat{m} = r_L\left[\ln\left\{1+B_\infty\left(1-\tau+q\frac{Y_{O\infty}}{\gamma}\right)\right\}\right] = \frac{r_L \ln(1+B)}{1-\frac{r_L}{r_f}} \tag{1.25 c}$$

$$\frac{r_f}{r_L} = 1 + \frac{\ln(1+B)}{\ln\left(1+\frac{Y_{O\infty}}{\gamma}\right)} \tag{1.25 d}$$

On the fuel side of the flame

$$T = \tau + \frac{1}{B_\infty}\left[e^{\hat{m}\left(\frac{1}{r_L}-\frac{1}{r}\right)} - 1\right] \tag{1.26 a}$$

and on the oxidizer side

$$T = e^{-\hat{m}/r} + \left(1-e^{\hat{m}/r}\right)\left(\frac{1}{B_\infty} - q - \tau\right) \tag{1.26 b}$$

Eqs. (1.25 a) and 1.25 b) define the conditions which must hold if a steady state is to exist. They define the droplet temperature τ if the saturated vapor line relation is known for the fuel. The formal limit of pure vaporization is obtained by passing $Y_{O\infty}$ to zero. Then $T_f = 1$ and $r_f = \infty$.

C. Convection and Spherical Symmetry

In actual combustors there is usually a strong convective field about a burning or vaporizing droplet[1]. In such a case the assumption of spherically symmetric burning in a quiescent atmosphere of infinite extent must completely break down. However, it is common to use the basic analytical

1. Typical Reynolds numbers for droplets in a rocket engine are Re <100.

treatment of the previous section while modifying it in only one respect; the outer boundary condition is not cast to infinity. There is no velocity considered tangential to the droplet surface within the diffusion film and spherical symmetry is still considered. An empirical correlation is then used to determine the thickness of the diffusion film. One such presently accepted relation is the Ranz and Marshall correlation (5)

$$\frac{\hat{m}\ \text{(with convection)}}{\hat{m}\ \text{(stagnant film)}} = 1 + 0.3(Pr)^{1/3} Re^{1/2}\ {}^{1} \qquad (1.27)$$

Such relations are usually used only for pure vaporization but may be extended to burning. They can be put into the form of Nusselt numbers for heat transfer and essentially are a measure of the quantity $r_f/(r_f - r_L)$ in Eq. (1.25 c). They have been used with some success in combustor performance calculations (6, 7).

Such an approach, while conceptually offensive, no doubt has its practical utility. The $Re^{1/2}$ term in the previously cited correlation strongly implies the existence of a boundary layer, and in reality there can be no definite edge to the diffusion film. This will provide a great deal of trouble if the same concepts are applied in an extension to the unsteady state where, again, artificial

1. This Reynolds number is based upon the droplet diameter and, although never stated, should be based on the free stream quantities (see p. 144).

constraints will have to be introduced.

Extensions of these concepts into the unsteady state have been attempted. Wieber and Mickelsen (8) used this convective correlation to determine instantaneous vaporization rates when a droplet was exposed to transverse acoustic oscillations of the ambient high temperature gas. Priem and Guentert (9) have used the same concepts to build a theory of combustion instability in liquid rocket engines. Two serious objections to these extensions can be raised: 1.) If the periodic variations in the ambient gas conditions are made transverse to the mean flow over the droplet or made across a stagnant droplet and if these oscillations are of sufficiently low amplitude, this cross flow is a low Reynolds number flow. In view of the above implication of a boundary layer the functional dependence of the convective correlation may seriously be in error. 2.) Frequently, the frequencies of oscillation which are considered are of the order of 1000 cps.[1] A calculation of the diffusion time within such a diffusion film for a typical droplet (150 microns in diameter, say) in a high temperature combustor would show that this time is of the same order of a cycle time. As is well known, when the characteristic times of two unsteady processes become

1. This is the order of frequencies observed in high frequency liquid rocket combustion instability.

commensurate, important unsteady effects may come into play. These quasi-steady convective correlations completely break down under such circumstances.

The effects of such unsteadiness and other types such as droplet heat-up, initial conditions relaxation, and droplet radius contraction form the subject matter which follows.

D. Summary

Under appropriate assumptions the basic equations of fluid mechanics have been written for the gas phase in a form convenient for all problems to be treated in the following pages. The common approach to droplet burning and vaporization theory has been presented. This steady state theory will be used both as an asymptotic limiting case in time and as a steady state theory about which time and space perturbation may be examined. Difficulties to be encountered in the time dependent treatments will ultimately result in complete rejection of the spherically symmetry treatment.

CHAPTER II

UNSTEADY ANALYSIS OF SIMPLIFIED MODELS

A. Heat-up, Boundary Movement, and Initial Conditions

In Chapter I, Eq's. (1.25 a and b) define a certain liquid temperature which must exist for the system to be in a steady state. This is the "wet bulb" temperature of the droplet. It is clear, however, that such a temperature will not exist in general immediately after droplet formation in an actual combustor. Obviously, there must be a transient heat-up period. Simultaneously, the conditions of the gas film surrounding a droplet are initially not those demanded by a quasi-steady state solution. There must therefore be a period of initial condition relaxation to the steady state condition. Finally, it is clear that there can never really be a steady state since the convective field changes as the droplet changes speed, and the droplet radius always contracts in time.

The above effects will be discussed with the aid of the spherically symmetric, convective, stagnant film model. The problem of droplet heat-up has been treated before (1, 6) always by assuming a quasi-steady gas film transient. Apparently this requires that the heat-up time be much larger than the gas film diffusion time. The standard treatments are always numerical; however, a quick analytical estimate for the characteristic time of heat-up is obtained in Appendix A. It is obtained under the following assumptions:

1. There is no vaporization during heat-up.
2. The Reynolds number between the ambient gas and droplet is constant during heat-up.
3. The droplet temperature is uniform within the droplet, implying a very large thermal conductivity (small diffusion time) or rapid internal circulation.

Of course, the quasi-steady spherical gas film is assumed. The result is that

$$t^*_{HU} = \left(\frac{\rho_L^* c_L^* r_L^{*2}}{3 \bar{\lambda}_g^*} \right) \left(1 - \frac{r_L^*}{r_f^*} \right) \left(\frac{T_{WB}^* - T_L(0)^*}{T_f^* - T_{WB}^*} \right) \tag{2.1}$$

The first grouping is a characteristic time, the second is an effect of the outer radius on the steepness of the temperature gradient at the droplet surface, and the third is the effect of the departure of the initial droplet temperature from the wet bulb temperature. A comparison of this time will be made with other process times later on.

Consider now the problem where the droplet heat-up time is very short and the primary unsteadiness is due to the droplet radius contraction and initial condition relaxation. Relaxing assumption 2 on p. 12 and taking the reference time, t_r^* , to be of the order of the droplet lifetime, t_d^* , the differential equations, initial conditions, and boundary conditions for the spherically symmetric time dependent problem are

$$\frac{1}{K}\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial \hat{m}}{\partial r} = 0 \tag{2.1 a}$$

$$\frac{\rho r^2}{K}\frac{\partial Y_K}{\partial t} + \hat{m}\frac{\partial Y_K}{\partial r} = \frac{\partial}{\partial r}\left(r^2\frac{\partial Y_K}{\partial r}\right) \tag{2.1 b}$$

$$\frac{\rho r^2}{K}\frac{\partial T}{\partial t} + \hat{m}\frac{\partial T}{\partial r} = \frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \tag{2.1 c}$$

$$\rho T = 1 \tag{2.1 d}$$

$$T(r, 0) = T_0(r) \tag{2.2}$$

$$T[r_L(t), t] = \tau \qquad T[r_f(t), t] = 1 \tag{2.3}$$

$$\left.\frac{\partial T}{\partial r}\right)_{r_L(t)} = \frac{\hat{m}[r_L(t)]}{r_L^2 B_\infty} = -\frac{1}{K_1 B_\infty}\frac{dr_L}{dt} \tag{2.4}$$

where K_1 is defined by

$$K_1 = \frac{\bar{\lambda}^* t_d^*}{\rho_L^* c_p^* r_{L_0}^{*2}} = K\,\frac{\rho_\infty^*}{\rho_L^*}$$

For the Lewis number equal to 1, the mass fraction equation, Eq. (2.1 b), is unnecessary; however, consideration of this equation and of other subsidiary conditions would have to be made for Le $\neq 1$. In particular there would be no justification for assuming τ to be constant in time in Eq. (2.3).

It is assumed that this non-linear set of equations of the parabolic type is well-posed under the imposed conditions, Eq's. (2.2) - (2.4). Note that since a total derivative appears at a constant temperature surface in Eq. (2.4), this is a non-linear boundary condition.

Briefly reviewing, what is in essence usually done to this set of equations is to consider all derivatives of order unity and consider K large. This assumes the ratio of a typical time in the problem, say the droplet lifetime, to a typical diffusion time is very large. Then all the terms containing time derivatives are neglected. Thus, from Eq. (2.1 a) $\hat{m}$ is constant in r and Eq. (2.1 c) becomes an ordinary differential equation for T which may be integrated subject to Eq's. (2.2). Eq. (2.4) may be used to find the interface position as a function of time. Another approach is to consider slow evaporation so that the convective term in Eq. (2.1 c) disappears. Then essentially the heat equation remains, but with a variable density which cannot be assumed constant unless Eq. (2.1 d) is abandoned. If the time derivative is also dropped here, essentially Laplace's equation remains and may be immediately integrated as before.

As pointed out by several investigators (10, 11), there are many things wrong with this approach. First, the problem is solved without reference to initial conditions and the solution is able to satisfy only the steady state conditions. Secondly, if the heat equation without the convective term is adopted, account is not given to important density variations and outward convective effects that occur in high temperature work. Finally, if Laplace's equation is adopted for the problem description and r_f cast to infinity, the droplet vapor content of the surroundings decreases from an infinite to a smaller infinite value as the droplet vaporizes.

Concerning the work which has been done on this and other related problems, Fuchs (10) obtained typical correction factors to describe the relaxation of the initial condition to the quasi-steady condition and the error made in the droplet lifetime due to the time dependence introduced by the contracting radius. This was, however, carried out for the heat equation and by approximate methods. Also, only one type of initial condition was considered, $T_o = 1$.

An exact solution for the heat equation under very general conditions of moving boundaries has been obtained by Kolodner (12). The solution is in the form of non-linear integro-differential equation for the interface position, if this is any simplification. The solution is based on the existence of the fundamental solution to the heat equation and uniqueness of the solution has been shown. In certain cases this work lends uniqueness to a great many other solutions as appear, for example, in Carslaw and Jaeger's work (13). For instance, in Neumann's problem of linear flow there is a liquid initially at temperature T and a solid initially at $x \leq 0$ which freezes into the liquid, the interface position given by X. Stating a condition that the temperature at x = 0 is constant for all time, a solution may be found where the interface position is given as $X/t^{1/2}$ = constant. Uniqueness for the problem is shown by Kolodner's work. This is a "similar" type solution and may be applied to many other problems with and without the convective term. Kirkaldy (11) has obtained an exact solution to the spherical problem with the convective term included but under the assumption that ρ is constant. It is found that if $r_L/t^{1/2}$ is constant the equation will reduce to an ordinary differential equation in the independent variable $\eta = r/t^{1/2}$. The objection here is that this procedure will not work for evaporation, but only condensation, because of the finiteness of r_L at $t = 0$. Also, density being constant is an intolerable assumption in the present work; finally these similar type solutions are only able to satisfy very specialized initial conditions.

There is one other interesting result obtained from these exact solutions. As expected, in a great many cases an appropriate expansion of the solution in a series will yield the quasi-steady solution as a leading term with the remaining terms important only near $t = 0$. Also, the interface motion is many times well approximated (sometimes exactly) by the quasi-steady solution. However, in the absence of an exact solution to the problem of interest, it still remains to investigate the error introduced by the quasi-steady theory. When $K \gg 1$ it is natural to attempt a regular expansion in the parameter $\frac{1}{K}$. However, the time derivatives are lost in the leading order set of differential equations. It is clear that this is a singular perturbation problem with wild behavior taking place near $t = 0$. What is then usually done is to attempt fo find an appropriate scale transformation of the independent and possibly dependent variables to retain the time derivatives in the leading order system of equations while, hopefully, obtaining a system amenable to exact analysis. Such a transformation has not been found.

Therefore, in Appendix A an alternate procedure has been adopted. The regular expansion in $\frac{1}{K}$ is carried out which yields a correction factor to the droplet radius to account for non-steady effects of the changing droplet size. It is valid only after a sufficient time has elapsed for relaxation of the initial conditions. Then under an admittedly inconsistent assumption that $\rho(r,t) = \rho(r,0)$ for all time a method developed by Frish (13) can be adapted to find a typical time estimate for the relaxation of the initial conditions. The correction factor obtained for the droplet radius is well behaved and, of course, of order $\frac{1}{K}$ for only two cases: 1.) The droplet must vaporize faster than the quasi-steady rate (r_L^2 decreasing linearly in time for r_f/r_L constant in time), and 2.) r_f cannot be cast to infinity. Thus, the correction is valid only for sufficiently high quasi-steady mass vaporization rates (see Eq. (A-19) of Appendix A). The estimate for the time required for initial condition relaxation is given by

$$t^*_{IC} = \left(\frac{\rho^* c_p^* r_{L_0}^{*2}}{\lambda^*}\right) \frac{\left(\frac{1}{r_f}-1\right)B}{\ln(1+B)} \int_1^{r_f} \rho(r') r'^2 \left[\frac{y^{(s)}(r')}{1+B\,y^{(s)}(r')}\right]\left[y^{(s)}(r') - y_0(r')\right] dr' \qquad (2.5)$$

where $y = \frac{T^*(r) - T_L^*}{T_f^* - T_L^*}$, $y^{(s)}$ is the quasi-steady distribution obtainable from Eq. (1.26a), and y_0 is the initial distribution at $t=0$. This integral has been numerically evaluated for $y_0 = 1$ and is tabulated in

Table 1. Again, an important observation is that r_f cannot be cast to infinity or the integral will blow up. More will be said concerning this in Section B of this chapter.

For convenience Appendix A uses t_r^* as the quasi-steady droplet lifetime and is repeated here for convenience for r_f/r_L constant.

$$t_d^* = \left(\frac{\rho_\infty^* \, c_p^* \, r_{L_0}^{*2}}{\lambda^*} \right) \left(\frac{\rho_L^*}{\rho_\infty^*} \right) \left(1 - \frac{r_f}{r_L}\right) \Big/ 2 \ln(1+B) \tag{2.7}$$

The typical gas phase diffusion time is

$$t_{dif}^* = \frac{\rho_\infty^* \, c_p^* \, r_{L_0}^{*2}}{\bar{\lambda}^*}$$

Therefore, the time ratios of interest are

$$t_1 = \frac{t_{Nu}^*}{t_d^*} \quad , \quad t_2 = \frac{t_{dif}^*}{t_d^*} \quad , \quad t_3 = \frac{t_{IC}^*}{t_d^*}$$

The first ratio is obtained from Eq's. (2.6) and (2.1)

$$t_1 = \left(\frac{T_{WB}^* - T_{L_0}^*}{T_f^* - T_{WB}^*} \right) \frac{c_L^*}{c_p^*} \, \frac{2}{3} \ln(1+B)$$

The result is primarily dependent upon the actual level of the wet bulb temperature. For volatile fuels or liquid oxygen the hope that $t_1 << 1$ is usually satisfied quite well for sufficiently high combustion temperatures. For example, take the case of ethyl alcohol burning with liquid oxygen in a combustor at 150 psia, Approximately for the alcohol,

$$t_1 \sim 10^{-1}$$

At a nominal Reynolds number of 40 based on the droplet diameter

$$t_2 \sim 1/60$$
$$t_3 \sim 10^{-3}$$

However, these numbers are heavily dependent on the actual chamber gas density and droplet Reynolds number. For any application a careful check should be made. In deriving the heat-up time the assumption was made that essentially $t_1/t_2 >> 1$. It is seen that while in this particular example the assumption may be reasonable, it may not be in other examples. However, it is believed that the order of magnitude of all the results is essentially predicted by the above developments even in view of many rough and sometimes inconsistent assumptions. Also, the dependences of these time ratios upon system parameter changes should be reasonably correct. The analysis above and in Appendix A attempts to separate effects although in reality the problem is a combined one, all processes taking place concurrently.

A further time ratio of interest is the liquid diffusion time to gas phase diffusion time; this will be reasonably important in the investigation of periodic solutions to the burning problem. In the present example

$$t_4 = (\rho_L^* c_L^* r_{L_0}^{*2} / \lambda_L^*) / t^*_{dif.} \sim 10^2$$

Note then that in this example assumption 3 on p. 21 is violated unless there is internal droplet circulation.

Note finally, for a smoothly changing convective field the unsteadiness due to a change in the effective r_f will also only be of order $\frac{1}{K}$ if the change takes place over a droplet lifetime.

The conclusions to be drawn are that in many pratical situations the heat-up time and initial condition relaxation times take place within a small fraction of the droplet lifetime. The unsteadiness due to droplet radius contraction is an effect of order $1/K$ and may often be considered negligible. Therefore, it may be reasonable to investigate the existence of a periodic solution to the burning problem when the ambient gas is undergoing acoustic oscillations while using the quasi-steady solution as the steady state solution and be assured that some measure of reality will be contained in the results.

B. A Difficulty in the Simple Approaches

It will be recalled that in two cases already treated r_f could not be cast to infinity without the results becoming meaningless. It should be observed that even in the quasi-steady treatment the ability to consider the outer boundary at infinity is a special property of spherical symmetry. Cylindrical symmetry (a cylindrical liquid surface) or a plane model would result in a zero burning or vaporization rate if the outer boundary were cast to infinity[1]. In the time dependent case even spherically symmetric models now give trouble. This trouble will again appear when a periodic solution is attempted for the spherically symmetric model; of course, it will also appear for the plane model. Therefore, somewhat artifical conditions must be imposed concerning this boundary in order to obtain well-posed problems.

This difficulty is a direct result of neglecting the actual convective field which must exist either by natural or forced convection; part of the physics is left out. It will be seen that even a plane liquid surface will yield a well-posed problem in the steady state through the well known phenomena of a boundary layer if proper account is taken of convection. The time dependent problem for any shape can also become well-posed if this part of the physics is included. Of course, proper treatment requires the abandonment of certain geometrical symmetries.

Still, certain interesting and informative results may be gained through the simpler geometries under

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1. Cylindrical symmetry would result in $\hat{m} = \rho v r \propto 1/\ln r_c$ where r_c is the outer boundary position specification. For a plane model $m = \rho v \propto 1/x_c$.

artificial boundary conditions. Therefore, before turning to the more complex problem some more attention will be given to the models with simple geometrical symmetry.

C. Periodic Solutions to Plane and Spherically Symmetric Models

It will now be considered that a well-defined steady state burning or vaporization process exists. That is, an infinite time has elapsed since the initial transients have started and the droplet or liquid surface does not contract in time. Section A of the chapter has attempted to evaluate the mathematically asymptotic sense in which such conditions may be fulfilled. Now it is assumed that acoustic oscillations of circular frequency ω^* are taking place in the ambient gas and an attempt will be made to find a solution of this same frequency to the burning problem. A first variation about the steady state will be considered; amplitudes will be arbitrarily small so that the perturbation problem will be linear in the perturbation amplitudes.

With the introduction of a new characteristic time, $1/\omega^*$, a further criteria to be introduced is that $t_d^* \omega^* \gg 1$ so that the interface position moves negligibly far in one cycle. This is somewhat equivalent to the statement $K >> 1$ since effects of frequency on the diffusion field will not be expected until ω^* becomes of the order of $1/t^*_{dif}$. It is best, therefore, to consider this as a "pourous plug" or "pourous sphere" problem. That is, the liquid is fed to the surface of a pourous metal of fixed dimensions at the appropriate mass flow rate to allow the liquid to just completely wet the plug surface at all times. The liquid body thus has a fixed dimension for all time. For real problems the asymptotic sense to which this is valid has just been given above. This is, in reality, a nearly periodic phenomena which is forced to be periodic in the limit. It is further assumed that in the unsteady state, no wave propagation phenomena is present in the diffusion film. That is, the cycle time is very long compared to the wave travel time through the film.

Consideration will be given to two problems. The first will be the influence of periodic oscillations on the vaporization rate from a plane liquid surface. As mentioned before, the outer edge of the diffusion film even in the steady state must be specified for this problem. An artificial boundary condition is therefore introduced; it is also continued into the unsteady problem where it is assumed that this position does not vary in time. This outer boundary may be roughly interpreted as a flame, edge of a vaporization "boundary layer", or some sort of artificial mass absorbing and releasing, heat releasing surface. If proper account would be taken of convection it is obvious that such a boundary would probably move in time. However, the plane symmetry, essential to analytic simplicity, would be lost. The oscillations at the edge of the film will not necessarily be taken as isentropic since they are oscillations which in a real case would exist still inside of a diffusion film. The second problem is concerned with periodic oscillations imposed "at infinity" over a burning droplet with spherical symmetry. Here a well defined flame boundary exists and it is initially hoped that a well-posed problem will emerge. No convection is considered, and the oscillations at infinity must be isentropic since no diffusion is present in a uniform field.

Consider first the plane problem; the coordinate system is shown in Figure 1b. All of the assumptions pertinent to the development of the spherical model in Chapter 1 are retained except that plane symmetry replaces spherical symmetry. The steady state solution is developed in Appendix B, as well as the periodic solution. A slight modification in the transport property law is used here as compared with assumption 4 on p. 12 of Chapter I. Although an average value is used for the steady state, this average value is allowed to vary in time under perturbation. Thus, the average value is assumed to vary as

$$\bar{\lambda}^* \propto T^{*\hat{\omega}}(x_f^*)$$

Then three particular problems are discussed. The first solves the problem where the temperature perturbation at the liquid surface is zero. This is the asymptotic limit of an infinite slope of the fuel partial pressure vs. temperature curve at the saturation line. It still retains the concept of an equilibrium boundary. Under such a situation there is no unsteady heat transfer to the liquid going towards heat-up rather than vaporization, and, also, the mass fraction equation is uncoupled from the energy equation. See Appendix B for the details. The second problem relaxes the above assumption When heat transfer becomes important to the liquid as far as heat-up is concerned new parameters which enter are the ratio of the gas phase diffusion time to the liquid phase diffusion time and the thermal conductivity ratio between the gas and liquid. For the example of Section A of this chapter these numbers are approximately 10^{-2} and 1/2, respectively. The third problem is the relaxation of the assumption of an equilibrium boundary. It is clear and well known that it is possible that mass may not be able to vaporize as fast as is demanded by the diffusion field calculation based upon an equilibrium boundary condition. What is essentially assumed normally is that the kinetics of evaporation are much faster than the diffusion speed. Evaporation kinetics have been recently reviewed (15) and will not be discussed at length here. It is sufficient to say that the driving force is the difference between the actual partial pressure of the vaporizing substance above the liquid and the equilibrium partial pressure at the droplet temperature. It is governed by an imbalance in the collision processes at the surface; that is, the emitted molecules exceed in rate those incident on the surface. Since this is a collision process, other parameters must enter, e.g., the molecular weight of the vaporizing substance and the "sticking probability" or "evaporation coefficient". However, to treat this problem a rather simple-minded assumption was used. Only perturbations in the mass fraction difference were considered. Therefore, the perturbation in the mass vaporization rate is

$$M_F^{(0)} = \overline{m}\left(y_{F\,\mathrm{equilibrium}} - y_{F\,\mathrm{actual}}\right)$$

$\overline{m}$ is calculated on the basis of a postulated steady state mass fraction drop.

The results for the plane model are summarized in Figures 2 - 6. These figures show the ratio of the instantaneous vaporization rate perturbation to the product of the steady state rate and the instantaneous pressure perturbation. This quantity is plotted vs. a frequency nondimensionalized by the film diffusion time, $(\rho_f^* c_p^* x_f^{*2})/\lambda^*$. As expected, significant response occurs when ω becomes $O[1]$.

Even in view of the extreme simplicity of this model the results show much of the physics involved in unsteady diffusion processes; many of the qualitative features are expected to carry over into a more complex model in Chapters III and IV. Figures 2-4 show the results for the first problem mentioned above. At the low frequency end the quasi-steady result is recovered. That is, the phase obviously goes to zero and the magnitude is $O[\gamma-1]$. This magnitude is extremely important in applications. It is of the order of the temperature perturbation which is depressed from the order of the pressure perturbation by $\gamma-1$, approximately. It is due to the fact that the quasi-steady state is dependent upon temperature sensitive functions, e.g., B and γ. For no convection the vaporization rate is almost pressure independent. As the frequency rises the temperature gradient at the surface increases and undergoes a maximum. This is characteristic of diffusion processes in the unsteady state. For instance, if the temperature were oscillated at one end of a solid slab the temperature gradient at the other end would undergo a maximum in a certain frequency range. However, this gradient would go to zero in the limit of infinite frequency; this is not the case for a compressible fluid. When the frequency becomes high enough a phenomenon known as the high frequency boundary layer comes into being. When a gas is compressed it attempts to heat-up instantaneously. However, if a boundary is maintained at a particular temperature $(\sigma(0) = \sigma_L = 0)$ heat diffusion must take place to smooth out the temperature field. At sufficiently low frequencies this can occur. But when the cycle time becomes shorter than the diffusion time this distribution cannot take place. Therefore, steep gradients take place near the non-natural boundary condition and diffusion is absent in the main part of the field. In fact, the heat transfer goes to infinity as the square root of frequency and the phase goes to $\pi/4$. The analytical result of a high frequency analysis is that for A = 0

$$\frac{M_F(0)}{\bar{m}\,\varphi} = \sqrt{i\omega}\;\frac{\gamma-1}{\bar{m}\,\tau\,\gamma} + O[1] + O\left[\frac{1}{\sqrt{i\omega}}\right] \tag{2.7}$$

The details of this analysis are not presented since they are a simplification of high frequency techniques to be used later on.[1] It is important to note that except in the transition region from low to high frequency

1. High frequency limits in this work are under the constraint that the wave propagation time in the film is still short compared to the cycle time. See p. 108.

behavior the response function is greater in magnitude for higher steady state mass flows. Another important result is that the phase is restricted within a narrow band and that a significant component of the response is in phase with the pressure.

Figure 5 shows the effect of considering the liquid heat transfer and temperature perturbation. The case with $\lambda_{L/g} \sqrt{\tau_4} = \sqrt{2}$ and b = 15[1] remains within ±5% of the case where these parameters are zero and infinity respectively. Therefore, this curve can also be used to compare with Figure 4 for the effect of γ and $\hat{\omega}$. It is seen that an area of additional response occurs for the rather severe condition $\lambda_{L/g} \sqrt{\tau_4} = 45$. This is apparently a coupling with the characteristic time of the liquid diffusion process. Over the main portion of the curve,however, the qualitative behavior is the same. Again, the vaporization rate perturbation will go to infinity as the square root of frequency but depressed in level since some heat transfer goes to the liquid.

Considering the effect of γ and $\hat{\omega}$, $\hat{\omega}$ is important only at very low frequency since it is an indicator of a temperature sensitive function. An increase

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1. b is the dimensionless slope of the fuel mass fraction vs temperature curve at the saturation line. See Appendix B.

in γ , however, generally raises the response by at least the percentage of increase in γ over the full range of frequency. Also, note an important effect in the high frequency limit, Eq. (2.7), directly showing the effect of compression work heating the gas.

The effect of finite evaporation kinetics is shown in Figure 6. For a rather severe but probable 10% mass fraction drop in the steady state the qualitative features remain the same, but, of course, the magnitude is depressed. Here, however, the "infinity" at high frequency will only appear if σ_L is held identically zero ($b = \infty$). For if $\sigma_L \neq 0$ some heat transfer must go to the liquid as $\sqrt{\omega}$ for $\omega \to \infty$. If σ_L and consequently the equilibrium value of $y_F(0)$ were to remain finite then the actual value of $y_F(0)$ would have to go as $\sqrt{\omega}$ to maintain the required vaporization rate perturbation. In such a case, however $y_F'(0)$ would go as ω saying the mass transfer rate was going as ω ; this is an obvious impossibility. On the other hand, if σ_L were to go to infinity as $\sqrt{\omega}$ all heat transfer would go to the liquid as ω which cannot be provided.

The true behavior must be that σ_L adjusts so that the component of heat transfer of $O[\sqrt{\omega}]$ is just absorbed by the liquid. What is left is a finite perturbation in the vaporization rate of $O[1]$. These problems will only appear, however, in the very high frequency range where other assumptions which have been made will also break down. It appears sufficient to say that over a reasonably wide frequency range ($0 \leq \omega \leq 40$) the qualitative behavior of the vaporization rate response is given correctly by neglecting evaporation kinetics.

One of the primary results of the plane model, then, is an estimate of the validity of certain simplifying assumptions. From now on the assumptions that $b = \infty$ ($\sigma_L = 0$) and $\mathcal{Y}_F(0)_{actual} = \mathcal{Y}_F(0)_{equilibrium}$ will be made with no further comment. They will become essential for simplified analytical treatment of more complex models.

It is now desirable to attempt to obtain a solution to the perturbed spherically symmetric problem in the hope that a well-defined flame boundary condition will emerge. In fact, a prime purpose is to find the perturbation in the mass burning rate at the flame since it may have important application to the field of unstable combustion. For that matter it is also probable that the vaporization rate perturbation is significant in this respect. Return, then, to the assumptions listed in Chapter I pertinent to the spherically symmetric model. Consider also the comments on the frequency level at the beginning of the discussion of the plane model. One further remark is now necessary; while the collapsed flame model may be applicable in the steady state, it may break down in the unsteady state. It must be further assumed that cycle times are much longer than typical chemical kinetics times. In fact, for the steady state model the assumption is that the chemical times are much shorter than diffusion times. Here, therefore, a restriction must exist on how short the cycle times can be made in relation to the diffusion times. The validity of the collapsed flame zone model will be discussed more fully for a more realistic geometrical configuration in Chapter III.

Appendix C contains the analytical development of the perturbed spherically symmetric model. Rather commonly, an expansion of the solution to periodic diffusion problems in powers of frequency is a convergent one. In fact, this was true for the plane model. It is shown, however, that this procedure diverges for the spherically symmetric model if the outer boundary is cast to infinity. In Section B of this chapter the reason for this was pointed out. In reality, no physically realistic spherically symmetric pressure oscillation can be provided unless there is some kind of spherically symmetric mass source and sink piston-like boundary. This is therefore not a well-posed problem for arbitrary frequency. However, the asymptotic case of high frequency can be treated because of the phenomenon of the high frequency boundary layer. If the condition at infinity is a natural one, i.e., an isentropic oscillation, a solution will exist in an asymptotic sense. Infinity is practically reached at a short distance from boundaries diffusion becomes localized within distances of $O[1/\sqrt{\omega}]$ of boundaries, the outermost boundary here being the flame. The details of this high frequency analysis are also contained in Appendix C. From this treatment it is also clear how the high frequency analysis was constructed for the plane model. Even a more detailed description will be given in Chapter IV.

The result in that once again the vaporization rate perturbation goes to infinity as the square root of frequency.

$$\frac{\hat{M}_F(r_L)}{\hat{\bar{m}}\,\varphi} = \frac{r_L^2}{\hat{\bar{m}}\,\tau^{1/2}}\,\frac{\gamma-1}{\gamma}\sqrt{i\omega} + O[1] \tag{2.8}$$

This should obviously be compared with Eq. (2.7). On the other hand an extremely striking result is that the <u>flame movement and mass consumption rate perturbations remain finite in the limit of infinite frequency</u>. Also, depending upon the sign, which has not been closely checked, this perturbation is real. That is, it is either in or 180° out of phase with the pressure.

The reason this finite behavior occurs is that the compression sets up a convective field which is natural for the boundary (flame); that is, the boundary conditions are automatically satisfied. The flame movement occurs (in the limit of infinite frequency) solely because the field is compressed. Note that this burning rate perturbation is made of three components: 1.) that due to an increase in the mass fraction gradient, 2.) the flame movement sweeping out the steady state mass flow, and 3.) the flame movement contracting the surface area. It is clear why the burning rate was not investigated with the plane model. It would have become infinite as $\sqrt{\omega}$ since the "flame" boundary was held stationary, a non-natural condition.

Briefly reviewing, with the aid of quite simple models under rather artificial boundary conditions some interesting results have been obtained concerning periodic oscillations of the droplet burning configuration. The physics contained in the results must in some degree carry over to a more complex model of the burning process. Therefore, encouraged by these results it remains to construct this more realistic model.

D. Summary

Conditions have been established whereby it is reasonable to investigate the existence of periodic solutions to the droplet burning problem. These conditions also evaluate the validity of the quasi-steady assumption in steady state droplet burning theory. Difficulties in the mathematical formulation of the unsteady burning problem have been pointed out. Finally, two geometrically simple configurations have been analysed and periodic solutions obtained under certain conditions. The physics contained in the results are sufficiently interesting to warrant the attempt at a solution to a more realistic and well-posed problem.

CHAPTER III

A CLASS OF CONVECTIVE DROPLET BURNING PROBLEMS: THE STEADY STATE

A. Statement of the Problem

In view of the difficulties encountered with simplified geometries as far as realistic results are concerned it is desirable to return to the full Navier-Stokes problem. In fact, consider a stationary liquid fuel body of arbitrary shape immersed in a uniform free stream flowing at a velocity, U_∞^*. The free stream contains an oxidizer of uniform mass fraction, Y_{o_∞}. A steady state is established whereby inward flowing and diffusing oxidizer meets outward flowing fuel at a thin flame zone which is wrapped about the leading edge of the body. The question of what is meant by a steady state has previously been discussed. The question of whether or not a flame can exist from the forward stagnation point will be discussed in Section C. It is desired to solve this problem for the flow field in the vicinity of the liquid body and to determine local vaporization and burning rates. Then consideration will be given to a perturbed problem by introducing a travelling isentropic sound wave in the free stream. This wave of arbitrarily small amplitude will travel parallel to the free stream flow and will be periodic in space and time. It will be assumed that the liquid shape does not oscillate in time.

The case of Reynolds number much greater than one will be treated. In actual combustors this is the case usually encountered during most of a droplet's life-time. It must be recognized, however, that the asymptotic procedure which will be used for large Reynolds number is going to be pushed quite far concerning a practical use. That is, it is hoped that reasonable quantitative information is given for Re as low as 10. The case of very large Re does not exist in practice either because droplet shattering will occur or because drag always keeps the relative speed between the ambient gas and droplet reasonably comparable.

It is clear that previous geometrical symmetry in two dimensions is lost. However, it will be assumed that the liquid body is either axi-symmetric at zero angle of attack or that the body is symmetric about an axis plane at zero angle of attack.

The free stream Mach number is to be small enough such that its square is negligible compared to unity. However, since strong temperature differences will necessarily appear, the steady state flow will not be considered incompressible.

B. Differential Equations and Boundary Conditions

Basically following Van Dyke (16), an intrinsic orthogonal coordinate system shown in Figure 1c is adopted.

The coordinate running along the liquid body from the forward stagnation point is s with corresponding velocity u. The normal to s is n with corresponding velocity v. The reference quantities of Chapter 1 are as follows:

$$t_r^* = a^*/U_\infty^* \qquad U_r^* = U_\infty^*$$
$$\rho_r^* = \rho_\infty^* \qquad T_r^* = T_\infty^*$$
$$x_r^* = a^* \qquad \mu_r^* = \mu_\infty^*$$

Therefore, K = 1, M_r is the actual free stream Mach number, M, and Re is the Reynolds number based upon droplet nose radius. Although this chapter is primarily concerned with the steady state, the time dependence and perturbation equationswill be developed here in order to avoid repetition in Chapter IV. Then with $\hat{j}=1$ for axi-symmetric flow or $\hat{j}=0$ for two-dimensional flow Eqs. (1.12)-(1.20) are written by usual vector relations where κ is the body curvature

Continuity

$$(1+\kappa n)(r+n\cos\theta)\rho_t + \left[(r+n\cos\theta)^{\hat{j}}\rho u\right]_s + \left[(1+\kappa n)(r+n\cos\theta)^{\hat{j}}\rho v\right]_n = 0 \tag{3.1}$$

Species Continuity

$$Re\left[Y_{K_t} + \frac{u}{1+\kappa n}Y_{K_s} + vY_{K_n}\right] = \frac{1}{\rho\, Sc}\Bigg\{(\mu Y_{K_n})_n + \mu Y_{K_n}\left[\frac{\hat{j}\cos\theta}{r+n\cos\theta} + \frac{\kappa}{1+\kappa n}\right] + \frac{1}{1+\kappa n}\left(\frac{\mu Y_{K_s}}{1+\kappa n}\right) + \frac{\hat{j}\,\mu Y_{K_s}}{(1+\kappa n)^2}\frac{(r+n\cos\theta)_s}{(r+n\cos\theta)}\Bigg\} \tag{3.2}$$

Longitudinal Momentum

$$\mathrm{Re}\left\{\rho\left[u_t + \frac{u u_s}{1+\kappa n} + v u_n + \frac{\kappa u v}{1+\kappa n}\right] + \frac{P_s}{1+\kappa n}\right\} =$$

$$\left[\mu\left(u_n + \frac{v_s - \kappa u}{1+\kappa n}\right)\right]_n + \frac{2}{1+\kappa n}\left[\mu\left(\frac{u_s+\kappa v}{1+\kappa n}\right)_s + \mu\left(\frac{2\kappa}{1+\kappa n}\right.\right.$$

$$\left.+ \frac{\hat{j}\cos\theta}{r+n\cos\theta}\right)\left(u_n + \frac{v_s-\kappa u}{1+\kappa n}\right) + \frac{2\hat{j}\, u(r+n\cos\theta)_s}{(1+\kappa n)(r+n\cos\theta)}\left[\frac{u_s+\kappa v}{1+\kappa n}\right.$$

$$\left.- \frac{u\,(r+n\cos\theta)_s}{(1+\kappa n)(r+n\cos\theta)} + \frac{v\cos\theta}{r+n\cos\theta}\right] + \frac{1}{1+\kappa n}\left\{\mu_2\left[\frac{u_s+\kappa v}{1+\kappa n}\right.\right.$$

$$\left.\left.+ v_n + \frac{\hat{j}}{r+n\cos\theta}\left(\frac{u}{1+\kappa n}(r+n\cos\theta)_s + v\cos\theta\right)\right]\right\}_s$$

(3.3)

Transverse Momentum

$$\mathrm{Re}\left\{\rho\left[v_t + \frac{u v_s}{1+\kappa n} + v v_n - \frac{\kappa u^2}{1+\kappa n}\right] + P_n\right\} = 2(\mu v_n)_n$$

$$+ \frac{1}{1+\kappa n}\left\{\mu\left[u_n + \frac{v_s-\kappa u}{1+\kappa n}\right]\right\}_s + 2\mu\left[\frac{\kappa}{1+\kappa n} + \frac{\hat{j}\cos\theta}{r+n\cos\theta}\right]v_n$$

$$- \frac{2\mu\kappa}{(1+\kappa n)^2}(u_s+\kappa v) - \frac{2\hat{j}\mu\cos\theta}{(r+n\cos\theta)^2}\left[\frac{u(r+n\cos\theta)_s}{1+\kappa n} + v\cos\theta\right]$$

$$+ \frac{\hat{j}\mu}{(1+\kappa n)(r+n\cos\theta)}\left(u_n + \frac{v_s-\kappa u}{1+\kappa n}\right)(r+n\cos\theta)_s$$

$$+ \left\{\mu_2\left[\frac{u_s+\kappa v}{1+\kappa n} + v_n + \frac{\hat{j}}{(r+n\cos\theta)}\left(\frac{u(r+n\cos\theta)_s}{1+\kappa n} + v\cos\theta\right)\right]\right\}_n$$

(3.4)

Energy

$$Re\left\{\rho\left[T_t + \frac{u\,T_s}{1+\kappa n} + v\,T_n\right] - M^2(\gamma-1)\left[\frac{u\,p_s}{1+\kappa n} + v\,p_n + p_t\right]\right\} =$$

$$\frac{1}{Pr}(\mu T_n)_n + \frac{1}{Pr\,(1+\kappa n)}\left(\frac{\mu T_s}{1+\kappa n}\right)_s$$

$$+ \frac{\hat{j}\,\mu T_s\,(r+n\cos\theta)_s}{Pr\,(1+\kappa n)^2\,(r+n\cos\theta)} + \frac{1}{Pr}\left(\frac{\kappa}{1+\kappa n} + \frac{\hat{j}\cos\theta}{r+n\cos\theta}\right)\mu T_n$$

$$+ \Phi$$

(3.5)

Dissipation Function

$$\Phi = \frac{M^2(\gamma-1)}{Re}\left\{\mu\left[2\left(\frac{u_s+\kappa v}{1+\kappa n}\right)^2 + 2v_n^2 + \frac{2\hat{j}}{(r+n\cos\theta)^2}\left(\frac{u(r+n\cos\theta)_s}{1+\kappa n}\right.\right.\right.$$

$$\left. + v\cos\theta\right)^2 + \left(u_n + \frac{v_s-\kappa u}{1+\kappa n}\right)^2\Bigg] + \mu_2\left[\frac{u_s+\kappa v}{1+\kappa n}\right.$$

$$\left.\left. + v_n + \frac{\hat{j}}{r+n\cos\theta}\left(\frac{u(r+n\cos\theta)_s}{1+\kappa n} + v\cos\theta\right)\right]^2\right\}$$

(3.6)

Longitudinal Diffusion

$$\frac{\mu}{Re\,Sc}\left(\frac{Y_{K_s}}{1+\kappa n}\right) = \rho u Y_K - \rho_K u_K \qquad (3.7)$$

Transverse Diffusion

$$\frac{\mu}{Re\, Sc} Y_{K_n} = \rho v Y_K - \rho_K v_K \tag{3.8}$$

State

$$\gamma M^2 p = \rho T \tag{3.9}$$

Transport Property Law

$$\mu = T^{\hat{\omega}} \tag{3.10}$$

Note in Eq. (3.10) that with $\mu_r^* = \mu_\infty^*$ a quite reasonable average law should result since $\tau^* < T_\infty^* < T_f^*$ in general.

The boundary conditions far away from the body must be

$$u(s,\infty,t) = 1 + \varepsilon \breve{U}(s,t) \tag{3.11a}$$

$$v(s,\infty,t) = \varepsilon \breve{V}(s,t) \tag{3.11b}$$

$$\rho(s,\infty,t) = 1 + \varepsilon \breve{\rho}(s,t) \tag{3.11c}$$

$$T(s,\infty,t) = 1 + \varepsilon \breve{T}(s,t) \tag{3.11d}$$

$$p(s,\infty,t) = \frac{1}{\gamma M^2} + \varepsilon \breve{p}(s,t) \tag{3.11e}$$

$$Y_0(s,\infty,t) = Y_{0\infty} \tag{3.11f}$$

The parts without ε are the steady state quantities; those including ε are perturbation quantities about the steady state, $\varepsilon << 1$. The perturbation functionsare yet to be defined and constructed.

The flame (sheet heat source) is defined as the location where inward flowing oxidizer meets outward flowing fuel in stoichiometric proportions and where the fuel and oxidizer mass fractions are zero. Since this is a pure heat source, continuity in mass flow, temperature, and the stress tensor must prevail across this discontinuity. Therefore, the boundary conditions at the flame which also define the flame position are

Mass Fraction

$$Y_o[s_f(t), n_f(t), t] = Y_F[s_f(t), n_f(t), t] = 0 \tag{3.12a}$$

Longitudinal Diffusion

$$\left.\frac{\mu}{Sc\,Re}\right)_f \left(\frac{1}{1+\kappa n_f(t)}\right) \left.Y_{F_s}\right)_f = -\left.\rho_F u_F\right)_f = \left.\frac{\rho_o u_o}{\dot{j}}\right)_f \tag{3.12b}$$

Transverse Diffusion

$$\left.\frac{\mu}{Sc\,Re}\right)_f \left.Y_{F_n}\right)_f = -\left.\rho_F v_F\right)_f = \left.\frac{\rho_o u_o}{\dot{j}}\right)_f \tag{3.12c}$$

Mass Continuity

$$\left.\rho u\right)_f, \quad \left.\rho v\right)_f \qquad \text{continuous} \tag{3.12d}$$

Normal Stress Continuity

$$\left. P\right)_f + \frac{\mu}{Re}\left\{(2\mu+\mu_2)\left[\left((r+n\cos\theta)\hat{\gamma}u\right)_s + \left((1+\kappa n)(r+n\cos\theta)\hat{\gamma}v\right)_n\right]\right\}\frac{1}{(1+\kappa n)(r+n\cos\theta)}\Bigg)_f \tag{3.12e}$$

is continuous

Shear Stress Continuity

$$\tau_{ij} \quad , j \neq i \qquad \text{is continuous} \tag{3.12f}$$

Temperature Continuity

$$T_f \qquad \text{is continuous} \tag{3.12g}$$

Longitudinal Heat Source

$$T_{s_{\text{fuel side}}} - T_{s_{\text{ox. side}}} + \frac{Pr}{Sc} q \left. Y_{F_s} \right)_f = 0 \tag{3.12h}$$

Transverse Heat Source

$$T_{n_{\text{fuel side}}} - T_{n_{\text{ox. side}}} + \frac{Pr}{Sc} q \left. Y_{F_n} \right)_f = 0 \tag{3.12i}$$

The following boundary conditions will be imposed at the liquid surface:

Velocity

$$u(s,o,t) = 0 \qquad v(s,o,t) = V_w(s,t) = V_{F_w}(s,t) \tag{3.13a}$$

Temperature

$$T(s,o,t) = \tau \quad = \text{constant} \tag{3.13b}$$

Mass Flow

$$\frac{\mu}{Re\,Sc} \left. Y_{F_n} \right)_w = \rho_w v_w Y_{F_w} - \rho_{F_w} v_w = m_w (Y_{F_w} - 1) \tag{3.13c}$$

$$\frac{\mu}{Re\,Pr} \left. T_n \right)_w = \frac{\rho_w V_w}{B_\infty} = \frac{m_w}{B_\infty} \tag{3.13d}$$

Mass Fraction

$$Y_F(s,0,t) = \overline{Y}_{F_w} + \varepsilon \breve{Y}_{F_w}(s,t) \tag{3.13e}$$

Eq. (3.13a) contains the no slip condition at the wall. This can, of course, only occur if the liquid viscosity is sufficiently high relative to that of the gas. There must be some slip velocity, in reality. However, the asymptotic procedure to be used in the solution of the problem will demand that u changes of order unity from the free stream to the boundary. This will, of course, always occur in the vicinity of the stagnation point. Therefore, other asymptotic procdures as used, for example, in liquid flow over gas bubbles (17) where the slip velocity is large must fail here. It is probably best to consider the no-slip condition as a "zeroth order" condition in an appropriate expansion in the slip velocity characterized by a small parameter.

The temperature condition, Eq. (3.13b), has been discussed at length before. These previous discussions also explain the mass fraction condition, Eq. (3.13e). The mass flow condition, Eqs. (3.13c) and 3.13d), are merely the transverse diffusion and heat transfer conditions at the wall. Note $v_{F_w} = v_w$ since the inert cannot move in an absolute sense at the wall.

Eq's. (3.1)-(3.13) represent the differential equations, algebraic conditions, and boundary conditions which specify the problem. Except for specification of the perturbation quantities in the free stream it is assumed that the system will admit a solution. The solution will be obtained by the method of inner and outer expansions. For Re >>1 it is presently believed correct that the expansion of any quantity, $f(s,n,t)$, in the form

$$f = \sum_{i=0}^{\infty} (\sqrt{Re})^{-i} f^{(i)}(s,n,t)$$

will yield the proper outer or essentially inviscid set of equations. The zeroth order yields the Euler equations and effects of viscosity and diffusion do not appear until second order. This set cannot, of course, satisfy the no slip boundary condition at the wall and still satisfy all the boundary conditions in the free stream since the highest derivative in Eq. (3.3) is lost in the zeroth order term. Indeed even the flame cannot exist in this region since no diffusion is present in the leading order terms. There must, therefore, exist a region of rapid transition in a layer close to the body, the inner region or boundary layer. For changes in u of order 1 through this layer it is presently believed correct that an expansion of the form

$$y = n\sqrt{Re}$$

$$u(s,y,t) = \sum_{i=0}^{\infty} (\sqrt{Re})^{-i} u^{(i)}(s,y,t)$$

$$v(s,y,t) = \sum_{i=0}^{\infty} (\sqrt{Re})^{-(i+1)} v^{(i)}(s,y,t)$$

$$T(s,y,t) = \sum_{i=0}^{\infty} (\sqrt{Re})^{-i} T^{(i)}(s,y,t)$$

$$p(s,y,t) = \sum_{i=0}^{\infty} (\sqrt{Re})^{-i} p^{(i)}(s,y,t)$$

$$Y_K(s,y,t) = \sum_{i=0}^{\infty} (\sqrt{Re})^{-i} Y_K^{(i)}(s,y,t) \qquad (3.14)$$

will yield the proper equations for the inner region. The time dependence in the problem is assumed independent of $\sqrt{Re}$ so that it may always be included in the leading order set of equations. Substituting Eq's. (3.14) into Eq's. (3.1)-(3.10) there results for the zeroth order system (the superscript$^{(o)}$ is left off for clarity) in the inner region

$$r^{\hat{j}} \rho_t + (r^{\hat{j}} \rho u)_s + (r^{\hat{j}} \rho v)_y = 0 \qquad (3.15a)$$

$$\rho\left[Y_{K_t} + u Y_{K_s} + v Y_{K_y}\right] - \frac{1}{Sc}(\mu Y_{K_y})_y = 0 \qquad (3.15b)$$

$$\rho\left[u_t + u u_s + v u_y\right] + p_s - (\mu u_y)_y = 0 \qquad (3.15c)$$

$$p_n = 0 \qquad (3.15d)$$

$$\rho\left[T_t + uT_s + vT_y\right] - \frac{(\mu T_y)_y}{Pr} - M^2(\gamma-1)(p_t + u p_s) - \Phi = 0 \qquad (3.15e)$$

$$\Phi = M^2(\gamma-1)\mu u_y^2 \tag{3.15f}$$

$$u = u_K \tag{3.15g}$$

$$\frac{\mu}{Sc} Y_{K_n} = \rho v Y_K - \rho_K v_K \tag{3.15h}$$

$$\gamma M^2 p = \rho T \tag{3.15i}$$

$$\mu = T^{\hat{\omega}} \tag{3.15j}$$

As pointed out by Van Dyke (16), μ_2 will not appear until second order.

Eq's. (3.15) are nothing more than the equations for the Prandtl boundary layer. Under the previous assumption that $M^2 << 1$, Φ is negligible and the term involving $u p_s$ in Eq. (3.15e) may be dropped. Terms involving the pressure are always confusing regarding this point since M^2 appears in the state equation, Eq. (3.15i). This should become clearer later on. The term involving p_t is kept since it will be later found $O[1/M]$ in the problem of interest.

The surface boundary conditions Eq's. (3.13), become to zeroth order

$$u(s,o,t) = o \tag{3.16a}$$

$$T(s,0,t) = \tau \tag{3.16b}$$

$$\frac{\mu}{Sc} Y_{F_y}(s,0,t) = m_w (Y_{F_w} - 1) \tag{3.16c}$$

$$\frac{\mu}{Pr} T_y(s,0,t) = m_w / B_\infty \tag{3.16d}$$

$$Y_F(s,0,t) = Y_{F_W}(s,t) \tag{3.16e}$$

It is clear that the flame must lie in this inner region since diffusion is present; the flame boundary conditions, Eq's. (3.12), become to zeroth order

$$Y_o(s_f, y_f, t) = Y_F(s_f, y_f, t) = 0 \tag{3.17a}$$

$$j\, Y_{F_y}\big)_f + Y_{o_y}\big)_f = 0 \tag{3.17b}$$

$$\rho u)_f\,,\; P,\; u_y)_f\,,\; T_f \qquad \text{continuous} \tag{3.17c}$$

$$T_y\big)_{\text{fuel side}} - T_y\big)_{\text{ox. side}} + \frac{Pr}{Sc}\, j\, Y_{F_y}\big)_f = 0 \tag{3.17d}$$

The expansion of Eq. (3.12b) is superfluous; ρv continuous and Eq. (3.12h) are not shown here because they represent higher order quantities in $1/\sqrt{Re}$ in the vector relations used to obtain them.

To patch the inner and outer expansions in the leading orders take an outer quantity $f_o^{(o)}(s,n,t)$ and rewrite it in inner variables $f_o^{(o)}(s, \gamma/\sqrt{Re}, t)$. Then expand about $Re = 0$ to obtain as a leading term $f_o^{(o)}(s,o,t)$ Similarly, an inner quantity is expanded to obtain $f_i^{(o)}(s,\infty,t)$. This is the classical boundary layer matching procedure. Therefore, the patching conditions for the boundary layer are exactly those given in Eq's.(3.11) except for Eq. (3.11b) which is automatically satisfied to the leading order in $1/\sqrt{Re}$. Then Eq's.(3.11), (3.15), (3.16) and (3.17) apparantly specify the boundary layer problem.

In principle, it is possible to go to higher orders with this method; that is, corrections may be obtained to the outer and inner flows and more terms of the expansion computed. However, for this particular problem there is a fundamental stumbling block. The outer inviscid flow being subsonic is elliptic in nature. For simple geometries the zeroth order solutions are well known. To solve for the first correction to this outer flow the boundary layer must be known <u>over the entire body</u> because of the elliptic nature of the outer flow. For closed bodies such as a sphere the expansion procedure to be used to gain such a solution is only asymptotically valid a short distance from the stagnation point. Also separation will occur. Therefore, to obtain a reasonable solution to the boundary layer over the full body is extremely difficult. It is possible to solve for the effects of longitudinal and transverse curvature of the body without reference to the outer flow correction. However, its elliptic nature comes in when the effect of the boundary layer displacement on the outer flow is considered. For some problems this effect may be negligible; however, here there is a highly blown boundary layer and there is no right to expect such a simplification.

The other problem, previously mentioned, is that of obtaining a valid solution over as much of the leading edge as possible. The Blasius series (18) may be used; it is an expansion of the solution in powers of s. However, it probably is only an asymptotic solution, not convergent in s. Therefore, for s of $O[1]$ or only approximately 60° from the forward stagnation the solution probably would become meaningless. This difficulty could most likely be circumvented by assuming, say, a parabolic body for which convergent series are known to result for other problems. At any rate, any of these techniques would become extremely cumbersome even in the steady state and a different philosophy has been adopted. Since it is now established that a boundary layer problem exists, the difference between different regions of the body lie primarily in the changing pressure gradient. Therefore, two problems will be solved; one will concern a zero steady state pressure gradient (the flat plate) and the second will concern the pressure gradient approapriate for stagnation point flow.

Before proceeding with the solution of this boundary layer problem the validity of the collapsed flame zone model will be discussed.

C. Validity of the Collapsed Flame Zone Assumption in High Reynolds Number Flow.

It is clear that if chemical reaction rates are not sufficiently fast the reaction zone may occupy a significant portion of the diffusion zone. If this occurs the peak temperature of the system must drop and temperature gradients must decrease. Breakdown of the entire flame may then occur since the heat transfer goes roughly as the peak system temperature but reaction rates decrease exponentially in temperature. This problem is critical in the region near the stagnation point in high velocity flow where the flame is expected to lie closest to the body surface. In fact some experimental results by Spalding (19) show conditions where breakdown occurs at the stagnation point and the flame moves into the wake, the droplet then acting as a flameholder.

An analytical bound on the accuracy of this assumption has been obtained by Brzustowski (20) for the spherically symmetric model. It will be here extended for the case of convective burning. It is clear that conditions on the chemical rates will be more stringent here since the significant parameter is the ratio of a nominal reaction zone thickness to the flame standoff distance from the body. For spherically symmetric burning r_f^*/r_L^* is usually $O[10]$. For a boundary layer $r_f/r_L = 1 + O[1/\sqrt{Re}]$ since the boundary layer thickness is $O[r_L^*/\sqrt{Re}]$ and the flame must lie within the boundary layer.

Assume that reaction takes place nominally in a distance from y_f to y_f'. Then in order for the fuel to burn a kinetic expression must hold in this zone. Assuming a homogeneous gas phase reaction mechanism applies and again that equal molecular weight among species prevails the fuel consumption rate per unit area is

$$m_F^* = W_F^{*(1-n)} \int_{y_f}^{y_f'} B(T)^* e^{-E^*/R^*T^*} \rho^{*n} Y_F Y_O^{(n-1)} dy^*$$

where n is the overall reaction order, E^* the activation energy, and $B^*(T)$ the pre-exponential frequency factor. Define appropriate average quantities such that

$$m_F^* = W_F^{*(1-n)} \bar{B}^*(T)\, e^{-E^*/R^* \bar{T}^*} \bar{\rho}^{*n} \bar{Y}_F \bar{Y}_0^{(n-1)} (y_f'^* - y_f^*) \qquad (3.18)$$

Now the accuracy of the theory is already limited by the boundary layer assumption to $O[1/\sqrt{Re}]$ compared to one. Therefore, in order for the reaction zone thickness not to appear

$$y_f'^* - y_f^* \leq O[y_f^*/\sqrt{Re}]$$

must be imposed. In the leading order for Y_F and Y_0 to be zero at the flame

$$Y_F\,,\, Y_0 \leq O[1/\sqrt{Re}]$$

must be imposed. This represents a deviation from Brzutowski's treatment. He assumed that Y_F and Y_0 must be less than or equal to the square of the small parameter under the argument that combustion could still take place in an atmosphere where Y_F and Y_0 were of the order of the small parameter. It is maintained here that this is an improper ordering of terms. Since the parameter itself can be made arbitrarily small the accuracy of the theory merely requires that a proper expansion would yield these quantities zero in the leading order. It is believed, therefore, that Brzustowski was unnecessarily strict.

Putting the inequalities into Eq. (3.18) and defining the dimensionless mass flow rate to be

$$m_F = \frac{m_F^*}{\rho_\infty^* u_\infty^*} = \frac{m_F^* a^*}{Re\ \mu_\infty^*}$$

there results

$$m_F \leq \frac{W_F^{*(1-n)}}{\mu_\infty^*} \bar{B}^* e^{-E^*/R^*\bar{T}^*} \bar{\rho}^{*n} a^{*2} \left(Re^{-1/2}\right)^{n+4} \qquad (3.19)$$

since y_f^* is $O[a^*/\sqrt{Re}]$ in a boundary layer. Now m_F must be $O[1/\sqrt{Re}]$ since $m_F \approx \rho v$ and v is $O[1/\sqrt{Re}]$ Therefore, rearranging Eq. (3.19)

$$a^{*2} \bar{\rho}^{*n} \geq Re^{\left(\frac{n+3}{2}\right)} \frac{\mu_\infty^* e^{E^*/R^*\bar{T}^*}}{W_F^{*(1-n)}\ \bar{B}^*} \qquad (3.20)$$

Converting the density to pressure by the perfect gas law, Eq. (3.20) becomes

$$a^{*2} p^{*n} \geq Re^{\left(\frac{n+3}{2}\right)} \frac{\mu_\infty^* e^{E^*/R^*T^*}}{W_F^{*(1-n)}\ \bar{B}^*} \left(\frac{R^* T^*}{W_F^*}\right)^n \qquad (3.21)$$

Eq. (3.21) states the minimum pressure below which the reaction zone thickness will not be sufficiently small as measured by the small parameter $Re^{-1/2}$ in relation to the diffusion zone thickness. Even in view of the relaxation of the strict condition of Brzutowski, a comparison with his result would show the right hand side of Eq. (3.21) much more strict in the presence of convection. For example, for n = 2 the effect of the small parameter is the fifth power for both theories. But the spherically symmetric theory would also have the factor $(r_L^*/r_f^*)^2$ on the right hand side. This is usually $O[10^{-2}]$. Consequently, the pressure limitation is increased by $O[10]$ in the convective theory. If the strict condition of Brzustowski had been employed here the pressure limitation would have been increased by $O[10\,Re]$ over his theory.

As an example, from Steacie (21) take $\bar{B} \approx 10^{14} \dfrac{cm^3}{\text{g-mole sec}}$ for n = 2 and also assume

$$\mu_\infty^* = 10^{-3} \text{ poise}$$
$$T^* = 3000\,°K$$
$$W_F^* = 30 \text{ g/g-mole}$$
$$E^* = 36 \text{ Kcal/g-mole}$$
$$Re = 50$$

Then

$$a^* p^* \geq 0.4 \text{ atm-cm}$$

Therefore, for a 100 μ droplet $p^* > 40$ atm. It is clear that high pressure combustors are implied for the collapsed flame model in a convective field This is, of course, only an extremely rough order of magnitude estimate. However, it shows that in practical application the existence of a flame from the forward stagnation point is open to question unless the pressures are quite high. From now on it will be assumed that these conditions are met and that still the pressure is sufficiently far from the critical pressure of the fuel.

There is an important subtlety in the above treatment. A calculation of the characteristic chemical kinetic time in the above example would be for $p^* = 40$ atm.

$$t^*_{CH} \approx \left[B^* e^{-E^*/R^*T^*} p^* W_F^* / R^* T^* \sqrt{Re}\right]^{-1} \approx 10^{-7} \text{ sec}$$

where $\sqrt{Re}$ enters since the order of the fuel mass fraction is $1/\sqrt{Re}$ in the reaction zone. The diffusion time is for $a^* = 10^{-2}$ cm.

$$t^*_{dif} = \frac{\rho^*_\infty a^{*2}}{\mu^*_\infty Re} \approx 10^{-5} \text{ sec}$$

A comparison of these times does not yield comparable numbers. Therefore, since this example is known to have a reasonably thick flame zone from the above theory, why aren't the times comparable? The reason is that the velocities of the reactants are extremely high in the reaction zone. Recall

$$\frac{\rho_F^* v_F^*}{\rho^*_\infty U^*_\infty} \approx \frac{m_F^*}{\rho^*_\infty U^*_\infty} \quad \text{is} \quad O\left[1/\sqrt{Re}\right]$$

But ρ_F^*/ρ^*_∞ is $O[1/\sqrt{Re}]$ in the reaction zone. Therefore, v_F^*/U^*_∞ is $O[1]$ whereas usually it is $O[1/\sqrt{Re}]$. Consequently, even though chemical times may be short, the reactants are moving quite rapidly and tend to spread out the reaction zone.

The above analysis is for the steady state. In the unsteady state characterized by a circular frequency it should be clear that the important quantity concerning the collapsed flame is the product Therefore, the cycle time can become quite a bit shorter than the diffusion time and the collapsed flame assumption will still hold if it holds in the steady state.

D. Construction of the Free Stream Perturbation Functions

A sound wave traveling parallel to the free stream flow and liquid body axis will be treated. The primary perturbation quantity will be chosen as the velocity. Therefore, in the free stream if $\breve{x}$ is the coordinate measured longitudinally from the forward stagnation point of the body, $\breve{u}$ in Eq. (3.11a) would become

$$\breve{u} = e^{i\omega^*\left(t^* + \frac{\delta \breve{x}^*}{\sqrt{\gamma R^* T^*_\infty} - \delta u^*_\infty}\right)} = \epsilon^{i\omega\left[t + \delta \breve{x}(M + O[M^2])\right]} \tag{3.22}$$

where the reference quantities are the same as in Section B. $\delta = \pm 1$ depending on whether the wave is traveling to the left or right. Since the theory will only be carried out linearly in ϵ, any wave shape can be made by superposition.[1] In particular a standing wave pattern may be made or a steep front traveling wave such as a shock wave followed by a decay can be constructed. It should be made very clear that what is desired is a periodic solution, one that neither grows nor decays in time. Therefore, initial conditions are not prescribed. The relaxation of actual initial conditions to the periodic solution should follow somewhat according to t^*_{dif} in order of magnitude. Also, this is a forced oscillation of a boundary layer and bears no relation to intrinsic boundary layer instability as investigated experimentally by Toong (22).

Eq. (3.22) is valid as it stands for a flat plate where s replaces χ since there is no interaction with the body and the wave. Therefore, for the flat plate the free stream velocity perturbation is

$$\breve{u}(s,\infty) = e^{i\omega\delta M s}\, e^{i\omega t} \tag{3.23}$$

For the stagnation point, however, there must be wave interaction with the body. This wave scattering problem is solved in Appendix D. The solution of the problem introduces some further restrictions and gives not only the perturbation form of the velocity at the stagnation point but the steady state boundary condition on u(s,0), here written in outer

1. It should be clear that this linear analysis excludes the the important second order phenomenon of "acoustic streaming" both in the free stream and boundary layer.

variables. A restriction on the unsteady problem is that the body is a sphere; this is not necessary for the steady state since the boundary condition yields

$$u(s,0) = \frac{3}{2}s + O[s^3] \tag{3.24}$$

which is, of course, the inviscid velocity for a true axi-symmetric stagnation point in the limit $\mathcal{r} = \infty$.

The result for the perturbation velocity in the free stream is

$$\breve{u} = \frac{3}{2}s\left\{1 + \frac{10}{9}i\delta\omega M + O[(\omega M)^2 + O[s^2]\right\}e^{i\omega t} \tag{3.25}$$

It is now required that $(\omega M)^2 << 1$; in view of the restriction on M^2 already made a restriction is therefore placed on the magnitude of ω . Eq. (3.25) presents a result which may have been contrary to physical intuition. If compressibility of the fluid is to be considered ($M \neq 0$) the quasi-steady velocity perturbation (u = 3/2 s) does not hold in the vicinity of the blunt stagnation point. The wave scattering is important.

Now with the perturbation forms for the velocity constructed it remains to construct consistent forms for the state variables. The unsteady Bernoulli Equation may be employed along the inviscid body streamline.

$$-\frac{\nabla p}{\rho} = \nabla\left(\frac{\vec{V}^2}{2}\right) + \vec{V}_t \tag{3.26}$$

For the flat plate this becomes in perturbation form at the edge of the boundary layer

$$i\omega \breve{\mathcal{U}} + \breve{u}_s = - \breve{p}_s \tag{3.27}$$

Integrating Eq. (3.27)

$$\breve{p} = \left[-\frac{\delta}{M} + O[1] \right] e^{i\omega(t+\delta M s)} \tag{3.28}$$

Since the oscillation is isentropic,

$$\breve{T} = \left[-\delta M(\gamma-1) + O[M^2] \right] e^{i\omega(t+\delta M s)} \tag{3.29}$$

and

$$\breve{\rho} = \left[-\delta M + O[1] \right] e^{i\omega(t+\delta M s)} \tag{3.30}$$

For the stagnation point

$$\breve{T} = -\delta M(\gamma-1) e^{i\omega t} \tag{3.31}$$

$$\breve{\rho} = -\delta M e^{i\omega t} \tag{3.32}$$

$$\breve{p} = -\frac{\delta}{M} e^{i\omega t} \tag{3.33}$$

To extract the pressure gradient perturbation for the stagnation point use Eq's. (3.25), (3.26), and (3.32) to obtain

$$\breve{p}_s = -\frac{3}{2} s \left\{ (3+i\omega) - \frac{3}{2}\delta M + \frac{10}{3} i\delta\omega M \left(1+\frac{i\omega}{3}\right) \right\} e^{i\omega t} \tag{3.34}$$

Finally, as previously stated a homocompositional field is always considered so that $\breve{Y}_o = 0$.

Note, that Eqs. (3.28) and (3.33) state that $\breve{P}_t$ is $O[1/M]$ which cannot be neglected in Eq. (3.15e). This term was left out in Illingworth's (23) work on the unsteady compressible boundary layer. It was precisely this term which was important in driving vaporization rate perturbations to infinity as $\sqrt{\omega}$ in the plane and spherically symmetric models; therefore, its importance is evident.

Since there are new parameters which enter the convective theory in the unsteady state it is desirable to gain an idea of the magnitudes involved. Consider first the frequency, $\omega = \omega^* a^*/U^*_\infty$. In a boundary layer a nominal thickness is (since this is how the normal coordinate is stretched) $\delta_B^* \sim a^*/\sqrt{Re}$. Therefore,

$$\omega = \frac{\omega^* \mu^*_\infty \rho^*_\infty a^{*2}}{\rho^*_\infty U^*_\infty a^* \mu^*_\infty} \sim \omega^* \left(\frac{\delta_B^{*2} \rho^*_\infty}{\mu^*_\infty} \right) \sim \omega^* \left(\frac{\rho^*_\infty c_p^* \delta_B^{*2}}{\lambda^*_\infty} \right)$$

and ω is clearly a ratio of a typical diffusion time transverse to the boundary layer to a cycle time, the same physical quantity previously encountered in simpler theories.[1] Consider the ωM product.

$$\omega M = \frac{\omega^* a^* U^*_\infty}{U^*_\infty \sqrt{\gamma R^* T^*_\infty}} \sim \frac{a^*}{L^*}$$

---

1. This is a controversial point in that a^*/U^*_∞ is more directly interpreted as a particle transit time. Nevertheless this interpretation will be adopted because of the behavior in the high frequency limit of a short diffusion penetration compared to the boundary layer thickness.

where L^* is the wavelength of the free stream disturbance. Therefore, ωM of $O[M]$ implies the droplet size of $O[ML^*]$ [1].

Note that if pressure had been referred to its steady state value rather than the dynamic head its order would have been M compared to 1 in the unsteady state while the velocity perturbation is $O[1]$. Therefore, the effect of the velocity oscillations, neglected in the simpler theories, can be expected to be quite important. The reason for this strange ordering is that in the unsteady state the velocities must propagate at primarily the sound velocity rather than the free stream velocity if M is low. Thus, in the unsteady state from the standpoint of the boundary layer velocity perturbations are raised by an order of magnitude of 1/M compared to what may have been intuitively believed.

Now attention will be turned to the solution of the unsteady boundary layer problem. Eq's. (3.15), (3.16), (3.17), (3.23), (3.25), and (3.28)-(3.33) are assumed to specify well-posed problems for both the stagnation point and flat plate. The steady state will be treated in the remainder of this chapter; the periodic solutions will occupy Chapter IV.

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1. For the flat plate, of course, "droplet size" must be interpreted as distance from the leading edge.

E. The Flat Plate - Steady State

As is usually the case it would appear that the flat plate case is the easiest to solve. It will be seen, however, that extreme difficulties arise in the unsteady case. In fact, the stagnation point being more difficult in the steady state becomes markedly less complex than the flat plate in the perturbed problem.

The time dependence will be carried through the development of the steady state equations so that independent development will not have to be made. To obtain the flat plate case from Eq's. (3.15) either set $\hat{j} = 1$ and $r =$ constant or take the two-dimensional case $\hat{j} = 0$. Then the governing equations under previous assumptions are

$$\rho_t + (\rho u)_s + (\rho v)_y = 0 \tag{3.35a}$$

$$\rho[u_t + u u_s + v u_y] = (\mu u_y)_y - p_s \tag{3.35b}$$

$$\rho[T_t + u T_s + v T_y] = \frac{1}{Pr}(\mu T_y)_y + M^2(\gamma-1) p_t \tag{3.35c}$$

$$\rho[Y_{K_t} + u Y_{K_s} + v Y_{K_y}] = \frac{1}{Sc}(\mu Y_{K_y})_y \tag{3.35d}$$

It will be assumed that $\hat{\omega} = Sc = Pr = 1$. Introducing a stream function, ψ , and the Howarth-Moore unsteady transform

$$\hat{\mu} = \int_0^y \rho(x,y',t)\,dy' \tag{3.36a}$$

$$x = s \tag{3.36b}$$

$$t = t \tag{3.36c}$$

$$u = \psi_{\hat{\mu}} \tag{3.36d}$$

$$v = -\frac{1}{\rho}\left(\psi_s + \hat{\mu}_t\right) \tag{3.36e}$$

continuity, Eq. (3.35a), is identically satisfied. Eq's. (3.35b)-(3.35d) become

$$\psi_{\hat{\mu}t} + \psi_{\hat{\mu}}\psi_{\hat{\mu}x} - \psi_x \psi_{\hat{\mu}\hat{\mu}} = -\frac{p_x}{\rho} + \gamma M^2 p\, \psi_{\hat{\mu}\hat{\mu}\hat{\mu}} \tag{3.37a}$$

$$\left(\frac{\partial}{\partial t} + \psi_{\hat{\mu}}\frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial \hat{\mu}}\right) T_{\hat{\mu}} = \gamma M^2 p\, T_{\hat{\mu}\hat{\mu}} + \frac{M^2(\gamma-1)}{\rho} p_t \tag{3.37b}$$

$$\left(\frac{\partial}{\partial t} + \psi_{\hat{\mu}}\frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial \hat{\mu}}\right) Y_{K\hat{\mu}} = \gamma M^2 p\, Y_{K\hat{\mu}\hat{\mu}} \tag{3.37c}$$

Basically following the methodology and nomenclature of Lam and Rott (24) and introducing the usual boundary layer variables,

$$\eta = \frac{\hat{\mu}}{\sqrt{2x}} \qquad x \to x \qquad t \to t \tag{3.38}$$

and the following perturbation forms

$$\psi = \sqrt{2x}\left[F(\eta) + \varepsilon\, P(\eta, x)\, e^{i\omega(t + \delta M s)}\right] \tag{3.39a}$$

$$T = \overline{T}(\eta) + \varepsilon\, \sigma(\eta, x)\, e^{i\omega(t + \delta M s)} \tag{3.39b}$$

$$Y_K = \overline{Y}_K(\eta) + \varepsilon\, y_K(\eta, x)\, e^{i\omega(t + \delta M s)} \tag{3.39c}$$

the steady state set of equations is first obtained from Eq's. (3.37)

$$F''' + F F'' = 0 \tag{3.40a}$$

$$\overline{T}'' + F\overline{T}' = 0 \tag{3.40b}$$

$$\overline{Y}_K'' + F\overline{Y}_K' = 0 \tag{3.40c}$$

Further introducing the unsteady variables

$$\eta \to \eta \qquad \xi = i\omega x \qquad t \to t \tag{3.41}$$

the perturbation set of equations valid to first order (or linear) in ε are obtained from Eq's. (3.37) and (3.39)

$$\begin{aligned} P_{\eta\eta\eta} + F P_{\eta\eta} + F''P - 2\xi[F'P_{\eta\xi} - F''P_\xi] \\ - 2\xi[F'P_\eta - F''P]\delta M - 2\xi P_\eta = \\ -2\xi(1+\delta M)T + \gamma\delta M F''' \end{aligned} \tag{3.42a}$$

$$\begin{aligned} \mathcal{L}[\sigma] = \left[\frac{\partial^2}{\partial\eta^2} + F\frac{\partial}{\partial\eta} - 2\xi F'\frac{\partial}{\partial\xi} - 2\xi(1+\delta M F')\right]\sigma = \\ -PT'(1+2\xi\delta M) - 2\xi P_\xi T' \\ + \gamma\delta M T'' + 2\xi\,\delta M(\gamma-1)T \end{aligned} \tag{3.42b}$$

$$\mathcal{L}[y_K] = -PY_K'(1+2\xi\delta M) - 2\xi P_\xi Y_K' + \gamma\delta M Y_K'' \tag{3.42c}$$

The bar superscript on steady state quantities has been deleted from the previous equations. The surface boundary conditions, Eq's. (3.16a), become in the steady state

$$F'(0) = 0 \tag{3.43a}$$

$$T(0) = \tau \tag{3.43b}$$

$$Y_F'(0) = -F(0)\,(Y_{F_W} - 1) \tag{3.43c}$$

$$T'(0) = -F(0)/B_\infty \tag{3.43d}$$

$$Y_F(0) = Y_{F_W} \tag{3.43e}$$

where it is clear that the mass flow at the surface is, from Eq's. (3.36), (3.38), and (3.39a),

$$\bar\rho\,\bar v(0) = -\bar\Psi_S(0,s) = -F(0)/\sqrt{2x} = \bar m_W \tag{3.44}$$

Therefore, the similar solutions in the variable η will require a vaporization rate decreasing as $1/\sqrt{x}$. The flame conditions, Eq's. (3.17), become

$$Y_0(\eta_f) = Y_F(\eta_f) = 0 \tag{3.45a}$$

$$j\,Y_F'(\eta_f) + Y_0'(\eta_f) = 0 \tag{3.45b}$$

$$F,\ F',\ F'',\ T \quad \text{continuous at } \eta_f \tag{3.45c}$$

$$T'(\eta_f -) - T'(\eta_f +) + q\,Y_F'(\eta_f) = 0 \tag{3.45d}$$ [1]

$$\eta_f = \frac{\hat{\mu}_f}{\sqrt{2x_f}} = \frac{1}{\sqrt{2x_f}} \int_0^{y_f} \rho(x_f, y')\, dy' \qquad (3.46)$$

The conditions in the free stream are obtained from Eq's. (3.11)

$$F'(\infty) = 1 \qquad (3.47a)$$

$$T(\infty) = 1 \qquad (3.47b)$$

$$Y_o(\infty) = Y_{o\infty} \qquad (3.47c)$$

Consider the solution of this set. The momentum equation, Eq. (3.40a), is the familiar Blasius equation and is uncoupled from the energy and species continuity equations except for the boundary conditions. For any value of the blowing parameter, F(0), the solution is unique and was obtained by Schlicting and Bussmann (25) long ago. A complication enters here because the blowing parameter is coupled with the other two equations by the surface transfer conditions. For pure vaporization work has been done by Spalding (26) concerning this problem. The new steady state modification, to the best of the author's knowledge, is the existence of a collapsed flame in this problem. At any rate, the problem is overdetermined as it stands; τ, Y_{F_W}, and B_∞ cannot all be assigned independently at the surface. There are eight boundary conditions for a seventh order system, otherwise. Note also, Y_o/j is the actual variable of interest if it is $\frac{Y_{O\infty}}{j}$ which is specified at infinity. Eq's. (3.45a), (3.45b), (3.47c), and (3.40c) are unaffected by this simplification.

As is well known, $T = a_K F' + b_K$ and $Y_K = c_K F' + d_K$ are integrals of Eq's. (3.40b and c) (the Prandtl integral) or that $T = \hat{a}_K Y_K + \hat{b}_K$ is an integral of Eq. (3.40b). It is a simple matter to determine the a's, b's, c's and d's through the boundary conditions, Eq's. (3.43), (3.45), and (3.47). Specifically,

$$
\begin{aligned}
a_F &= (T_f - \tau)/F_f' & b_F &= \tau \\
a_O &= \frac{T_f - 1}{F_f' - 1} & b_O &= T_f + \frac{Y_{Fw}\, j}{Y_{O\infty}}(T_f - 1) \\
c_F &= Y_{Fw}/-F_f' & d_F &= Y_{Fw} \\
c_O &= Y_{O\infty}/(1 - F_f') & d_O &= -j\, Y_{Fw} \\
\hat{a}_F &= (\tau - T_f)/Y_{Fw} & \hat{b}_F &= T_f \\
\hat{a}_O &= (1 - T_f)/Y_{O\infty} & \hat{b}_O &= T_f
\end{aligned}
\qquad (3.48)
$$

Therefore, application of Eq's. (3.45b and d), and (3.43c and d) together with Eq's. (3.48) yields the following important relationships:

$$Y_{Fw}/(1 - Y_{Fw}) = B_\infty (T_f - \tau) \qquad (3.49a)$$

$$T_f = \left(1 + q\frac{Y_{O\infty}}{j} + \tau \frac{Y_{O\infty}}{j\, Y_{Fw}}\right) \Big/ \left(1 + \frac{Y_{O\infty}}{j\, Y_{Fw}}\right) \qquad (3.49b)$$

$$Y_{F_w}/(1-Y_{F_w}) = -F(0)\,F_f'\,/\,F''(0) \tag{3.49c}$$

$$(1-F_f')/F_f' = Y_{O_\infty}/j\,Y_{F_w} \tag{3.49d}$$

Eq's. (3.49a and b) together with the equilibrium saturated vapor line relation for the fuel define the wet bulb temperature of the droplet and the flame temperature. They are identical in form to Eq's. (1.25a and b) and are not restricted to flat plate flow. They take care of the redundant boundary condition mentioned above and specify conditions necessary for a steady state. These relations occur for any flow in which heat and mass transfer are similar (Pr = Sc) and will occur in any boundary layer flow under this restriction. Eq's. (3.49c and d) are peculiar to this problem since they rely on the integrals of T and Y_K as linear functions of F'. Note then that F" (0)/F(0) and F' are functions only of the two parameters Y_{O_∞}/j and Y_{F_w}. In fact, the momentum equation can be solved by specification of these two quantities alone; they fix the value of the blowing parameter F(0). Note that when Y_{F_w} = 1, separation occurs. Therefore, the assumption often made in the literature that the liquid temperature is the saturated temperature for the total pressure of the gas leads to an inconsistency in this respect. Note, furthermore, that a most interesting quantity is the mass burning rate at the flame which by Eq. (3.12c) and tracing through all the transformations is

$$\sqrt{2x}\ \overline{m}_{F_f} = -Y_F'(\eta_f) = \frac{Y_{F_W} F_f''}{F_f'} \tag{3.50}$$

which is also specified by only the two parameters Y_{F_W} and Y_{O_∞}/j. Therefore, without knowing the details of the complete temperature and mass fraction fields a great deal of information may still be extracted. The pure vaporization limit is taken as $Y_{O_\infty}/j \to 0$. The Blasius results are obtained as Y_{F_W} and $\frac{Y_{O_\infty}}{j} \to 0$ but maintaining $Y_{O_\infty}/j\ Y_{F_W}$ finite.

Desirous of obtaining solutions for many values of the parameters and because of the need of the steady solution for the unsteady integration, the Blasius equation, Eq. (3.40a), subject to Eq's. (3.43a), (3.47a), and (3.49 cand d) has been numerically integrated. For an initial guessed F"(0) the equation is integrated out to η = 8.5 by a Milne variable step size technique contained in a package subroutine available for the IBM 7090 computing machine (27). The value of F' (∞) is compared with 1 and the initial values are assumed linear in the final values; Newton's method is used to guess new initial values and eventually converge. Convergence is quite rapid for the Blasius equation since, as is well known, this equation behaves well about $\eta = \infty$ All numbers are believed accurate to at least .005%. The results are presented in Table 2 and Figures 7 and 8. Quantities at the boundaries have been considered more important than detailed profiles. Therefore, only in

Figure 8 are samples given of actual profiles.

There are several interesting points to note on Figure 7. First, the mass burning rate is usually of the order of one half of the vaporization rate, at least for large Y_{F_w} and moderate Y_{O_∞}/j. Therefore, approximately half of the vaporized mass is convected downstream rather than burned. Secondly, as expected, an increase in $\frac{Y_{O_\infty}}{j}$ moves the flame position toward the liquid surface and captures more of the vaporized mass at the burning surface. Therefore, just as in spherically symmetric theory, $\frac{Y_{O_\infty}}{j}$ has a weak effect on the vaporization rate for fixed Y_{F_w} but here it has an extremely strong effect on how much is locally burned rather than convected downstream. Third, a change in Y_{F_w} for fixed Y_{O_∞}/j has a marked effect on F(0) but almost no effect on $\bar{m}_{F_f}$ especially at high Y_{F_w} This is because of the opposing effect that a high surface rate and high Y_{F_w} "blow" the flame out to a larger η_f, thereby capturing less of the vaporized mass. Finally, note that F"(0) is a unique curve, independent of Y_{F_w} and Y_{O_∞}/j. This is obvious, since the Blasius equation will only admit one solution for each F(0).

Finally, in order to fix an idea concerning correspondence of this convective theory and the old spherically symmetric convective theory note the definition

$$\overline{m} = \frac{\bar{\rho}^* \bar{v}^{(0)*}}{\rho_\infty^* u_\infty^*} = \sqrt{Re} \ \frac{\bar{\rho}^* \bar{v}^*}{\rho_\infty^* u_\infty^*} = \frac{1}{\sqrt{Re}} \left(\frac{\bar{\rho}^* \bar{v}^* a^*}{\mu_\infty^*} \right) \tag{3.51}$$

The last grouping is the non-dimensional mass flow of the simpler theories. Since $\overline{m}$ is $O[1]$ here, the mass flow referenced to (μ_∞^*/a^*) is proportional to $\sqrt{Re}$ as, of course, is known for heat transfer in laminar flow. Comparing with the correlation, Eq. (1.27), it is seen that here for $Re \to 0$ there is no vaporization while there was for spherically symmetric burning. This is a fault of the asymptotic procedure being used since errors are being made of $O[1/\sqrt{Re}]$ compared to $O[1]$; this error becomes infinite as $\sqrt{Re} \to 0$. The above comments are not, of course, restricted to the flat plate.

F. The Stagnation Point - Steady State

At the stagnation point r = s where, at least for a sphere, the error made is $O[s^3]$. For $\hat{j} = 1$ the governing equations from Eq's. (3.35) are

$$\rho_t s + (s\rho u)_s + (s \rho v)_y = 0 \tag{3.52a}$$

$$\rho \left[u_t + u u_s + v u_y \right] = (\mu u_y)_y - P_s \tag{3.52b}$$

$$\rho \left[T_t + u T_s + v T_y \right] = \frac{1}{Pr} (\mu T_y)_y - M^2 (\gamma - 1) P_t \tag{3.52c}$$

$$\rho\left[Y_{K_t} + u Y_{K_s} + v Y_{K_y}\right] = \frac{1}{Sc}\left(\mu Y_{K_y}\right)_y \tag{3.52d}$$

Again it will be assumed that $\hat{\omega} = Sc = Pr = 1$. First using the Mangler transform

$$\begin{aligned}
\xi &= s y \\
z &= \int_0^s s^2 ds = s^3/3 \\
\tau &= t \\
u &= u \\
\rho \hat{v} &= \frac{1}{s}\left(\rho v + \frac{y}{s}\rho u\right)
\end{aligned} \tag{3.53}$$

Eq's. (3.52) become

$$\rho_\tau / s^2 + (\rho u)_z + (\rho \hat{v})_\xi = 0 \tag{3.54a}$$

$$\rho\left[\frac{T_\tau}{s^2} + u T_z + \hat{v} T_\xi\right] = (\mu T_\xi)_\xi + M^2(\gamma - 1)\frac{P_\tau}{s^2} \tag{3.54b}$$

$$\rho\left[\frac{u_\tau}{s^2} + u u_z + \hat{v} u_\xi\right] = (\mu u_\xi)_\xi - P_z \tag{3.54c}$$

$$\rho\left[\frac{Y_{K_\tau}}{s^2} + u Y_{K_z} + \hat{v} Y_{K_\xi}\right] = (\mu Y_{K_\xi})_\xi \tag{3.54d}$$

Then using a modified Howarth-Moore transform

$$\begin{aligned}
\hat{\mu} &= \int_0^\xi \rho \, d\xi \\
x &= z \\
t &= \tau \\
u &= \psi_{\hat{\mu}} \\
\hat{v} &= -\frac{1}{\rho}\left(\psi_z + \frac{\hat{\mu}_\tau}{s^2}\right)
\end{aligned} \tag{3.55}$$

Eq's. (3.54) become

$$\rho\left[\frac{\psi_{\hat{\mu}t}}{s^2}+\psi_{\hat{\mu}}\,\psi_{\hat{\mu}x}-\psi_x\,\psi_{\hat{\mu}\hat{\mu}}\right]=\rho\left(\rho\mu\,\psi_{\hat{\mu}\hat{\mu}}\right)_{\hat{\mu}}-P_x \tag{3.56a}$$

$$\frac{T_t}{s^2}+\psi_{\hat{\mu}}\,T_x-\psi_x\,T_{\hat{\mu}}=\gamma M^2 p\,T_{\hat{\mu}\hat{\mu}}+\frac{M^2(\gamma-1)}{\rho\,s^2}P_t \tag{3.56b}$$

$$\frac{Y_{K_t}}{s^2}+\psi_{\hat{\mu}}\,Y_{K_x}-\psi_x\,Y_{K\hat{\mu}}=\gamma M^2 p\,Y_{K\hat{\mu}\hat{\mu}} \tag{3.56c}$$

Then introducing the boundary layer variables

$$\eta=\frac{3}{2}\,\frac{\hat{\mu}}{s}\qquad x\to x\qquad t\to t \tag{3.57}$$

and the following perturbation forms

$$\psi(\eta,x,t)=(3x)^{2/3}\left[F(\eta)+\varepsilon\,P(\eta)\,e^{i\omega t}\right] \tag{3.58a}$$

$$T(\eta,t)=\bar{T}(\eta)+\varepsilon\,\sigma(\eta)\,e^{i\omega t} \tag{3.58b}$$

$$Y_K(\eta,t)=\bar{Y}_K(\eta)+\varepsilon\,y_K(\eta)\,e^{i\omega t} \tag{3.58c}$$

the steady state set is first obtained from Eq's. (3.56).

$$F'''+\frac{4}{3}FF''+\frac{2}{3}\left(\bar{T}-F'^2\right)=0 \tag{3.59a}$$

$$\bar{T}''+\frac{4}{3}F\bar{T}'=0 \tag{3.59b}$$

$$\bar{Y}_K''+\frac{4}{3}F\bar{Y}_K'=0 \tag{3.59c}$$

Noting that, miraculously, a set of ordinary differential equations is obtained for the perturbation set, there results

$$P''' + \frac{4}{3} F P'' - \frac{4}{3} F' P' + \frac{4}{3} P F'' - \frac{4}{9} i\omega P' = \gamma \delta M F'''$$
$$- \frac{4}{9} \left\{ \bar{T} \left[3 + i\omega + \frac{3}{2} (\gamma - 1) \delta M \right] + \frac{3}{2} \sigma + \frac{10}{3} i\omega \delta M \bar{T} \left(1 + \frac{i\omega}{3} \right) \right\} \tag{3.60a}$$

$$\mathcal{L}_2[\sigma] = \left[\frac{d^2}{d\eta^2} + \frac{4}{3} F \frac{d}{d\eta} - \frac{4}{9} i\omega \right] \sigma = -\frac{4}{3} P \bar{T}'$$
$$+ \frac{4}{9} \delta M i\omega (\gamma - 1) \bar{T} + \gamma \delta M \bar{T}'' \tag{3.60b}$$

$$\mathcal{L}_2[y_K] = -\frac{4}{3} P \bar{Y}_K' + \gamma \delta M \bar{Y}_K'' \tag{3.60c}$$

The last term in brackets in Eq. (3.60a) is the perturbation in (p_K / ρ) and is readily obtained from Eq's. (3.34) and(3.31) since $\bar{p}_s = -\frac{9}{4} s$.

Again tracing through the transformations, Eq's. (3.57), (3.55), and (3.53), the surface boundary conditions in the steady state are obtained from Eq's. (3.16a).

$$F'(0) = 0 \tag{3.61a}$$

$$T(0) = \tau \tag{3.61b}$$

$$Y_F(0) = Y_{F_w} \tag{3.61c}$$

$$Y_F'(0) = \frac{4}{3} F(0) (1 - Y_{F_w}) \tag{3.61d}$$

$$T'(0) = -\frac{4}{3} F(0) / B_\infty \tag{3.61e}$$

In this problem numerical factors somewhat cloud the picture. The surface mass flow is, from the transformation equations

$$\bar{m}_w = \bar{\rho}\,\bar{v})_w = s\,\bar{\rho}\,\hat{v})_w = -s\,\bar{\Psi}_z(o,s) = -2F(o) \qquad (3.62)$$

which is independent of x. The flame and infinity boundary conditions are given by Eq's (3.45) and (3.47). The flame position is given from Eq's. (3.57), (3.53), and (3.55).

$$\bar{\eta}_f = \frac{3}{2}\,\frac{\hat{\mu}_f}{s_f} = \frac{3}{2}\int_0^{\bar{y}_f} \bar{\rho}\, dy' \qquad (3.63)$$

which is, in general, a function of s.[1]

The solution of the steady state set, Eq's. (3.59) subject to Eq's. (3.61), (3.45), and (3.47), is not as simple as for the flat plate due to the fact that the energy and momentum equations are coupled because of the non-zero pressure gradient. Therefore, at least one immediate integral is lost. The analytic information lost is that equivalent to Eq's. (3.49c and d) in the flat plate case. However, once again heat and mass transfer are similar from Eq's. (3.59b and c), and, as previously mentioned, Eq's. (3.59 and b), still hold since $T = \hat{a}_K Y_K + \hat{b}_K$ where $\hat{a}_K$ and $\hat{b}_K$ are determined in Eq's. (3.48).

Two pieces of information have been lost since the integral of a second order equation has been lost. These are $F''(0) = f[Y_{F_w}, Y_{O_\infty}/j, F(o)]$ and $F_f' = f[Y_{F_w}, Y_{O_\infty}/j]$.

1. However, it is not here. In fact, although disguised by the many transformations, all state variables and v are functions of y alone to $O[s^2]$. This is expected from well known stagnation point solutions. The fact that ordinary differential equations result for the unsteady case comes from this property, although there is no a priori knowledge that this should be so.

They are replaced by the specification of two additional parameters, here taken as τ and q. However, because of the nature of the lost information the numerical integration becomes more difficult; an iteration must be carried out for both F(0) and F"(0). In this problem the mass burning rate at the flame is given by

$$\bar{m}_{F_f} = -\frac{3}{2}\bar{Y}_F'(\bar{\eta}_f) = -\frac{3}{2}\frac{\bar{T}'}{\delta_F}(\eta_f-) \tag{3.64}$$

It should again be pointed out that no longer is the solution of the momentum equation independent of the energy equation, therefore, the presence of the heat source alters the velocity profile. As such it is definitely impossible to obtain the solution from previous work on the compressible boundary layer with blowing, Reshotko's (28) work, for example. The limit of pure vaporization is still $Y_{O_\infty} \to 0$. But to obtain standard blown incompressible boundary layer results it is now necessary to also pass $\tau \to 1$. To obtain non-blown, incompressible theory (Homann flow) $q \to 0$ and $Y_{F_W} \to 0$ with Y_{F_W}/Y_{O_∞} finite. Once again separation occurs when $Y_{F_W} = 1$ since an infinite evaporation rate is implied.

The numerical integration proceeds as for the flat plate except that now a two-fold iteration takes place, F(0) and F"(0) vs $F'(\infty)$ and $T(\infty)$. Again the initial values are assumed linear in the final values for the iteration procedure. There is more difficulty in convergence here, since, as is well known, this set of equations contains one solution about infinity which blows up powerly in η. The initial values guessed must be "sufficiently" close to the final values for convergence to be obtained. If it was obtained, only a maximum of five iterations were necessary for convergence within .00001 of the final values.

The results are shown in Table 3 and Figures 9-11. Figure 11 is the only detailed profile calculation presented since the primary interest is in the quantities at the boundaries, to prevent a further iteration in the time dependent case. This particular figure shows the interesting phenomenon of the velocity overshoot near the heat source due to the pressure gradient acting on a fluid of low density (high temperature).

Fixing τ and q, Figure 9 should be compared with Figure 7 for the flat plate. The conclusions are similar to those of the flat plate. The primary differences are the fact that η_f is smaller at a comparable Reynolds number since, obviously, the boundary layer thickness is smaller. Then note that F"(0) is no longer a unique curve with F(0) since it now depends on the temperature field which is influenced by Y_{∞}/j and Y_{F_w}. A surprizing result is that, for fixed Y_{F_w}, F"(0) increases with F(0) whereas normally the higher the blowing parameter the less the skin friction. This is due to the strong effect that Y_{O_∞}/j has on increasing T_f and decreasing η_f so that the velocity overshoot is greater and moved closer to the body thereby increasing the velocity gradient. Extremely strange behavior is exhibited at sufficiently low Y_{F_w} where for fixed Y_{O_∞}/j, F"(0) vs F(0) undergoes a maximum (see Table 3). Finally note, with the use of Eq. (3.64), that approximately one-half of the vaporized mass is again convected downstream rather than burned. However, more is captured for a comparable case than for the flat plate. Figure 10 shows the effect of the two additional parameters, γ and τ. The results are somewhat obvious on physical grounds except that the dependences are quite weak. The exception is F"(0) vs τ, the strong dependence of which occurs through the viscosity law, $\hat{\omega} = 1$.

G. Summary

A formulation of the droplet burning problem in a high Reynolds number flow has been given. Treatment of the Prandtl boundary layer results as a natural consequence of an appropriate asymptotic expansion of the full Navier-Stokes equation. Difficulties in obtaining a solution over the entire droplet have been indicated; the decision to treat two typical boundary layer cases was made. An investigation of the collapsed flame zone assumption in a high Reynolds number flow has been made; the results show that the existence of such a flame is marginal for normal pressures. The formulation for the perturbation problem for the stagnation point and flat plate has been presented. Finally, the steady state solution to these two problems has been obtained. Some of the results of the simpler theories of burning have been shown to carry over into the convective case. However, one of the most important results has been shown to be the fact that much of the vaporized mass is convected into the wake rather than burned in a diffusion flame surrounding the droplet.

CHAPTER IV

PERIODIC SOLUTIONS TO THE CONVECTIVE DROPLET BURNING PROBLEM

A. The Boundary Conditions

Immediately upon moving to the solution of Eq's. (3.42) and (3.60) an important subtlety should be noted. From Eq's. (3.38) and (3.36a) note that the boundary layer variable η is a time dependent variable with a steady state component and a perturbed component,

$$\eta = \bar{\eta} + \varepsilon \breve{\eta}$$

This occurs since ρ contains such components. Therefore, $F(\eta)$, $\bar{T}(\eta)$, and $\bar{Y}_k(\eta)$ are not truly steady state unless $\eta = \bar{\eta}$. Something has been left out in the construction of Eq's. (3.40), (3.42), (3.59), and (3.60), the perturbation in η. This distinction was omitted in Illingworth's work (23) but made no consequence in the results. It will be important here because of the existence of a boundary condition in the interior of the boundary layer (the flame). To take care of this difficulty two approaches can be taken. The first would be to actually consider perturbations in the independent variable, and the steady state equations, Eq's. (3.40) and (3.59), would be correct for $F(\eta)$, $\bar{T}(\eta)$, $\bar{Y}_k(\eta)$. However the perturbation equations, Eq's. (3.42), are then incorrect since no terms appear to account for

$\breve{\eta}$. The second procedure, and the one that will be adopted is to analytically continue any quantity $\mathcal{F}(\eta)$ into the complex η-plane as $\mathcal{F}(\eta) = \mathcal{F}(\bar{\eta}) + \varepsilon \breve{\eta}\, \mathcal{F}'(\bar{\eta})$. Then the total quantity of interest is

$$\mathcal{F}(\eta) = \bar{\mathcal{F}}(\bar{\eta}) + \varepsilon \left[\breve{\mathcal{F}}(\bar{\eta}) + \breve{\eta}\, \bar{\mathcal{F}}'(\bar{\eta}) \right] + O[\varepsilon^2]$$

As long as the continuation is valid the solution of Eq's (3.40) and (3.59) will yield $\bar{\mathcal{F}}(\bar{\eta})$ and $\bar{\mathcal{F}}'(\bar{\eta})$. In this case the perturbation equations, Eq's. (3.42) and (3.60) are correct. The two procedures must be equivalent to first order in ε.

Then consider the evaluation of any quantity $\mathcal{F}$ at the flame. At a fixed x position

$$\mathcal{F}(x_f, \eta_f(s_f, y_f(t), t), t) = \bar{\mathcal{F}}(x_f, \eta_f(\bar{y}_f)) + \varepsilon \left[\breve{\mathcal{F}}(x_f, \bar{\eta}_f) + \breve{y}_f\, \eta_y)_f\, \bar{\mathcal{F}}_\eta)_f \right]$$

$$= \bar{\mathcal{F}}(\bar{\eta}_f) + \varepsilon \Big\{ \breve{\eta}_f(\bar{y}_f)\, \bar{\mathcal{F}}'(\bar{\eta}_f) + \breve{\mathcal{F}}(x_f\, \bar{\eta}_f, t) + \breve{y}_f(t)\, \bar{\mathcal{F}}'(\bar{\eta}_f)\, \eta_y)_f \Big\}$$

Therefore, define

$$\tilde{\eta}_f = \breve{\eta}_f(\bar{y}_f) + \breve{y}_f \frac{\partial \eta}{\partial y}\Big)_f \qquad (4.1)$$

and

$$\mathcal{F}(x_f, \eta_f, t) = \bar{\mathcal{F}}(\bar{\eta}_f) + \varepsilon \left[\tilde{\eta}_f\, \bar{\mathcal{F}}'(\eta_f) + \breve{\mathcal{F}}(\eta_f, x_f, t) \right] + O[\varepsilon^2]$$

$\breve{\eta}_f$ is the true perturbation in the flame position in η - space, accounting for movement of the upper limit of integration and the changing integrand in Eq. (3.36a) Then $\mathcal{J}$ evaluated at the flame consists of a steady state part plus a perturbation quantity $\breve{\mathcal{J}}$ plus a continued part of the steady state term, but here continued to the true flame position. Normally, for any other position within the layer the term involving $\breve{y}_f$ would not appear in Eq. (3.51). Now the flame boundary conditions, Eq's. (3.17), may be cast in perturbation form

$$y_o + \tilde{\eta}_f \bar{Y}_o' = y_F + \tilde{\eta}\, \bar{Y}_F' = 0$$

or

$$y_o + j\, y_F = 0 \tag{4.2a}$$

$$y_{o_\eta} + j\, y_{F_\eta} = 0 \tag{4.2b}$$

$$\sigma(\eta_f -) - \sigma(\eta_f +) + g\, y_F = 0 \tag{4.2c}$$

$$\sigma_\eta(\eta_f -) - \sigma_\eta(\eta_f +) + g\, y_{F_\eta} = 0 \tag{4.2d}$$

Eq's.(3.42a) and (3.60a) continuous across the flame (4.2e)

where use has been made of the steady state boundary conditions, Eq's. (3.45). Condition (4.2e) requires also that P_ξ and $P_{\eta\xi}$ are continuous across the flame which would follow from a higher order investigation of Eq's. (3.12d and f).

The surface conditions, Eq's. (3.16), become

$$P_\eta(\xi,0) = 0 \tag{4.3a}$$

$$\sigma(\xi,0) = 0 \tag{4.3b}$$

$$y_F(\xi,0) = y_{F_w}(\xi) \tag{4.3c}$$

$$\sigma_\eta(\xi,0) = -\gamma\delta M\frac{F(0)}{B_\infty} - \frac{1}{B_\infty}\Big[(1+2\xi\delta M)P + 2\xi P_\xi\Big]_w \tag{4.3d}$$

$$\begin{aligned} y_{F_\eta}(\xi,0) = {} & \gamma\delta M F(0)(1-Y_{F_w}) - y_{F_w}F(0) \\ & + (1-Y_{F_w})\Big[(1+2\xi\delta M)P + 2\xi P_\xi\Big]_w \end{aligned} \tag{4.3e}$$

for the flat plate. For the stagnation point Eq's. (3.53d and e) are replaced by the following:

$$\sigma'(0) = -\frac{4}{3}\gamma\delta M\frac{F(0)}{B_\infty} - \frac{4}{3}\frac{P(0)}{B_\infty} \tag{4.3f}$$

$$y_F'(0) = \frac{4}{3}(1-Y_{F_w})\Big[\gamma\delta M F(0) + P(0)\Big] - \frac{4}{3}y_F(0)F(0) \tag{4.3g}$$

In Eq's. (4.3d and g) the first term is due to the perturbation in the $\rho\mu$ product. It therefore accounts in part for compression and heating of the boundary layer.

The conditions at infinity become from Eq's. (3.22), (3.28), and (3.11f) for both the stagnation point and flat plate

$$P_\eta(\xi,\infty) = 1 \tag{4.4a}$$

$$\sigma(\xi,\infty) = -\delta M(\gamma-1) \tag{4.4b}$$

$$y_0(\xi,\infty) = 0 \tag{4.4c}$$

B. The Flat Plate - Low Frequency

Lam and Rott (24) have shown that for the incompressible problem an expansion of the solution in powers of (2ξ) is a convergent one although, of course, not practically useful for high frequency. Convergence has not been investigated nor will be claimed here. However, such an expansion will be attempted and will at least provide asymptotic information for low frequency. Therefore, assume

$$\begin{aligned} P &= \sum_{n=0}^{\infty} (2\xi)^n g^{(n)}(\eta) \\ \sigma &= \sum_{n=0}^{\infty} (2\xi)^n h^{(n)}(\eta) \\ y_K &= \sum_{n=0}^{\infty} (2\xi)^n k_K^{(n)}(\eta) \end{aligned} \tag{4.5}$$

Substituting Eq's (4.5) into Eq's. (3.42) and collecting like powers of frequency

$O[(2\xi)^0]$

$$g^{(0)\prime\prime\prime} + F g^{(0)\prime\prime} + F'' g^{(0)} = \gamma\delta M F''' \tag{4.6a}$$

$$\mathcal{L}_1[h^{(0)}] = \left[\frac{d^2}{d\eta^2} + F\frac{d}{d\eta}\right] h^{(0)} = -g^{(0)} T' + \gamma\delta M T'' \tag{4.6b}$$

$$\mathcal{L}_1[k_K^{(0)}] = -g^{(0)} Y_K' + \gamma\delta M Y_K'' \tag{4.6c}$$

$O[(2\xi)']$

$$g^{(1)'''} + F g^{(1)''} + 3F'' g^{(1)} - 2 g^{(1)'} = -(1+\delta M)T + g^{(0)'} + \delta M[F' g^{(0)'} - F'' g^{(0)}] \tag{4.7a}$$

$$\mathcal{L}_1[h^{(1)}] = 2F' h^{(1)} + (1+\delta M) h^{(0)} + \delta M(\gamma - 1)T - 3 g^{(1)} T' - \delta M\, g^{(0)} T' \tag{4.7b}$$

$$\mathcal{L}_1[k_K^{(1)}] - 2F' k_K^{(1)} = (1+\delta M)\, k_K^{(0)} - 3 g^{(1)} Y_K' - \delta M\, g^{(0)} Y_K' \tag{4.7c}$$

$O[(2\xi)^n]\ ;\ n \geq 2$

$$g^{(n)'''} + F g^{(n)''} - 2n\, g^{(n)'} F' + 2n\, g^{(n)} F'' = g^{(n-1)'} + \delta M\,[F' g^{(n-1)'} - g^{(n-1)} F''] \tag{4.8a}$$

$$\mathcal{L}_1[h^{(n)}] - 2n F' h^{(n)} = (1+\delta M F') h^{(n-1)} - (2n+1) g^{(n)} T' - \delta M\, g^{(n-1)} T' \tag{4.8b}$$

$$\mathcal{L}_1[k_K^{(n)}] - 2n F' k_K^{(n)} = (1+\delta M F') k_K^{(n-1)} - (2n+1) g^{(n)} Y_K' - \delta M\, g^{(n-1)} Y_K' \tag{4.8c}$$

The boundary conditions, Eq's. (4.2)-(4.4), can be similarly expanded. At first, a troublesome point looks evident, that is, the arbitrariness of δM . Both the sign and magnitude depend upon the particular problem of interest. This term appears due to the travelling nature of the wave and therefore through the pressure, pressure gradient, and associated state variables under oscillation. Since these terms would not appear for an incompressible fluid, M $\equiv$ 0, they will be called terms due to a "pressure effect". The remaining terms would still appear for an incompressible fluid and will be called terms due to a "velocity effect".[1] It is possible, since these terms only appear in the inhomogeneous parts of Eq's. (3.56)-(3.58) and, what is equivalent, in the boundary conditions, to separate the two effects by assuming

$$g^{(n)} = g^{(n)}_{(0)} + \delta M\, g^{(n)}_{(1)}$$

$$h^{(n)} = h^{(n)}_{(0)} + \delta M\, h^{(n)}_{(1)}$$

$$k_K^{(n)} = k_{K_{(0)}}^{(n)} + \delta M\, k_{K_{(1)}}^{(n)} \qquad (4.9)$$

The problem may then be solved for arbitrary δM.

Consider the problems one at a time. The differential equations for the $^{(0)}_{(0)}$ problem are

$$g^{(0)\,'''}_{(0)} + F g^{(0)\,''}_{(0)} + F'' g^{(0)}_{(0)} = 0$$

$$\mathcal{L}_1[h^{(0)}_{(0)}] = -g^{(0)}_{(0)} T'$$

$$\mathcal{L}_2[k_{K\,(0)}^{(0)}] = -g^{(0)}_{(0)} Y_K'$$

with the boundary conditions from Eq's. (4.3)

$$g^{(0)\,'}_{(0)}(0) = 0$$

$$h^{(0)}_{(0)}(0) = 0$$

$$k_{F\,(0)}^{(0)}(0) = k_{F_W\,(0)}^{(0)}$$

$$h^{(0)\,'}_{(0)}(0) = -g^{(0)}_{(0)}/B_\infty$$

$$k_{F\,(0)}^{(0)\,'}(0) = k_{F_W\,(0)}^{(0)} F(0) + (1 - Y_{F_W})\, g^{(0)}_{(0)}(0)$$

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1. The naming of these effects also stems from their properties explained at the bottom of p. 138.

From Eq's. (4.4)

$$g_{(0)}^{(0)\prime}(\infty) = 1$$
$$h_{(0)}^{(0)}(\infty) = 0$$
$$k_{o\,(0)}^{(0)}(\infty) = 0$$

and conditions at the flame stand as in Eq's. (4.2) if quantities are replaced by the quantities. This is nothing but the quasi-steady velocity problem for which the following solution is easily constructed:

$$g_{(0)}^{(0)} = \frac{F + \eta F'}{2}$$
$$h_{(0)}^{(0)} = a_K \frac{F''\eta}{2} \qquad (4.10)$$
$$k_{K\,(0)}^{(0)} = c_K \frac{F''\eta}{2}$$

The a's and c's are defined in Eq's (3.48). This solution satisfies the condition $k_{F_W\,(0)}^{(0)} = 0$. This solution is clear since all that is being perturbed is the free stream velocity in the Reynolds number. Since the steady state mass flow is proportional to $\sqrt{Re}$ a perturbation should yield $g_{(0)}^{(0)}(0) = F(0)/2$, which it does. Nothing else is disturbed in the field, the boundary layer thickness just oscillates.

However, the quasi-steady pressure effect is more difficult since compression of the layer is occuring and complex boundary conditions occur. Using Eq's. (4.9), (4.6), (4.3), and (4.4) the $\binom{(0)}{(1)}$ problem is specified by

$$g_{(1)}^{(0)\prime\prime\prime} + F g_{(1)}^{(0)\prime\prime} + F'' g_{(1)}^{(0)} = \gamma F''' \qquad (4.11a)$$

$$\mathcal{L}_1[h_{(1)}^{(0)}] = -a_K g_{(1)}^{(0)} F'' + \gamma a_K F''' \qquad (4.11b)$$

$$\mathcal{L}_1[k_{K(1)}^{(0)}] = -c_K g_{(1)}^{(0)} F'' + \gamma c_K F''' \qquad (4.11c)$$

under the boundary conditions

$$g_{(1)}^{(0)\prime}(0) = 0 \qquad (4.12a)$$

$$h_{(1)}^{(0)}(0) = 0 \qquad (4.12b)$$

$$k_{F(1)}^{(0)}(0) = y_{Fw(1)}^{(0)} \qquad (4.12c)$$

$$h_{(1)}^{(0)\prime}(0) = -\frac{\gamma F(0)}{B_\infty} - g_{(1)}^{(0)}/B_\infty \qquad (4.12d)$$

$$k'^{(0)}_{F(1)}(0) = \gamma F(0)(1-Y_{Fw}) - y_{F(1)}^{(0)} F(0) + (1-Y_{Fw})\, g_{(1)}^{(0)}(0) \qquad (4.12e)$$

$$g_{(1)}^{(0)\prime}(\infty) = 0 \qquad (4.12f)$$

$$h_{(1)}^{(0)}(\infty) = 1-\gamma \qquad (4.12g)$$

$$k_{o(1)}^{(0)}(\infty) = 0 \qquad (4.12h)$$

and Eq's. (4.2) written for the ${}^{(0)}_{(1)}$ components. Because of the surface mass transfer no simple solution can be constructed from the steady state solution as was the case for the quasi-steady velocity effect. A machine integration is required. Here a three-fold iteration is required for the quantities $y_{F_w(1)}^{(0)}$, $g_{(1)}^{(0)\prime\prime}(0)$, and $g_{(1)}^{(0)}(0)$ as functions of Eq's. (4.12g and h). The same type of iteration technique was used as for the steady state; it was also used in all calculations to follow.[1] The integration procedure has switched, however, to a Runge-Kutta technique using the same package routine (27) as before. This switch was made for all time-dependent calculations to save computing time during the initial iteration steps partly due to the fact that all time-dependent momentum equations have one unstable solution about $\eta = \infty$ The interval size used was 0.01 and now the error is more or less on an absolute basis rather than percentage-wise. This error is estimated from the difference in the steady state quantities based on the previous procedure, since they were also integrated along with the unsteady quantities to avoid reading in a table of values. All time dependent quantities are believed accurate to .0001.

The additional parameters which must be specified for the problem are τ, g, and γ, since integral information is not available. Note that $\frac{k_0}{j}$ is the actual variable of interest and that j does not have to be independently specified. This can be seen from Eq's. (4.11c), (4.12h), (3.48), and (4.2a and b). A quantity of extreme interest is the perturbation in mass burning rate at the flame. Expanding as in Eq. (4.5) and evaluating the perturbation from of Eq. (3.15h) at the flame,[2]

1. This iteration is, however, exact since due to the linearity of the equations the final values are linear in the initial values.
2. All quantities of this kind are perturbation quantities divided by ε and, hence, finite.

$$-\sqrt{2x}\, M_{F_f} = -\sqrt{2x} \sum_{n=0}^{\infty} (2f)^n \left[M_{F_{f(0)}}^{(n)} + \delta M M_{F_{f(1)}}^{(n)} \right] = \left[y_{F_\eta} + \tilde{\eta} \bar{Y}_F'' - \gamma \delta M \bar{Y}_F' \right]_f \qquad (4.13)$$

$\tilde{\eta}_f$ has a similar expansion and is evaluated from Eq. (4.2a). The $\sqrt{2x}$ dependence is extracted by dividing by the steady mass burning rate in Eq. (3.50). Such normalized quantities have much more physical meaning than the absolute number, anyway. Note from Eq. (4.13) the three components making up this perturbation. One is due to the perturbed mass fraction gradient, one is due to the flame movement "sweeping out" the steady state field, and the third is due to compression and heating of the boundary layer changing the "steady state" gradients and transport properties. The results for the quasi-steady pressure effect are shown in Figures 12-14. In viewing these figures the following points should be borne in mind:

1. Except for $g_{(1)}^{(0)}$, a negative quantity means an increase in phase with a pressure increase.[1]
2. $g_{(1)}^{(0)} > 0$ signifies a vaporization rate increase in phase with a pressure increase.[2]

Figure 12 shows the dependence on the parameters of interest in the steady state, Y_{F_W} and Y_{O_∞}/j. The following conclusions can be drawn:

1. As expected, when the vaporization rate increases, the shear stress decreases.
2. In analogy to the plane case of Chapter 2 $g_{(1)}^{(0)}(0)$ and $g_{(1)}^{(0)}(0)/\bar{m}_w\sqrt{2x}$ increase for higher steady state mass flows only in

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1. This is a result of the form of Eq's. (3.28) and (3.33).
2. This is a consequence of Eq. (3.36e).

the curve vs Y_{F_w} [1].

3. The small values of $y_{F_w(1)}^{(0)}$ again insure that $\sigma(0) \approx 0$ is a reasonable assumption.
4. $M_{F_f(1)}^{(0)} / \bar{m}_{F_f}$ is a decreasing function with increasing steady state vaporization and burning rates.
5. The trends of $\tilde{\eta}_{f(1)}^{(0)}$ primarily determine the trends of $M_{F_f(1)}^{(0)} / \bar{m}_{F_f}$ showing the extreme importance of flame movement.
6. The quasi-steady vaporization and burning rate perturbations are much higher than for the plane model due to interaction with the convective field. In fact, they are $0[1]$ rather than $0[\gamma-1]$, in general.
7. All quantities except the shear stress increase in phase with the pressure.

The primary conclusion to be drawn from Figures 13 and 14 is that dependences on the additional parameters $\mathfrak{g}$ and τ are weak especially for usual values of these parameters ($\mathfrak{g} > 6$, $\tau < 0.5$). The dependence on γ is strong, however, as will be shown later through high frequency analyses and for the stagnation point.

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1 The normalized function $g_{(1)}^{(0)}(0) / \bar{m}_w \sqrt{2x}$ is not shown but can be readily calculated from Table 2.

Once again, for clarity, the analogous functions for the quasi-steady velocity effect may be computed from Eq's. (4.10) and are presented here

$$g^{(0)}_{(0)}(0) = F(0)/2$$

$$M_{F_{f(0)}}^{(0)}/\bar{m}_{F_f} = 1/2$$

$$g^{(0)\prime\prime}_{(0)}(0) = \frac{3}{2}F''(0)$$

$$\widehat{\tau}^{(0)}_{(0)} = -\bar{\tau}_f/2$$

$$y_{F_{w(0)}}^{(0)} = 0 \tag{4.14}$$

All are effects in phase with the velocity.

The problem for first order in 2ξ is now attacked. However, only the $\binom{(1)}{(0)}$ problem will be solved - that for the velocity effect. The reason is that the primary interest is in the magnitudes in phase with the pressure. Because the expansion parameter is $i\omega x$ any quantity of first order concerning the pressure effect must necessarily be 90° out of phase with the pressure. For a travelling wave this is also true with the velocity effect, but recall that a standing wave can be constructed from two of the travelling waves being considered. Since a standing wave is characterized by a 90° phase shift between the velocity and pressure, any quantity which is 90° out of phase with the velocity is of interest. This result which can be directly seen from the expansion procedure is true for all boundary layer work along this line: Unsteady pressure effects in phase with the pressure do not become important until $O[\omega^2]$.

The $\substack{(1)\\(0)}$ problem is specified by Eq's. (4.7), (4.9), (4.3), and (4.4) to be

$$g^{(1)\,'''}_{(0)} + F g^{(1)\,''}_{(0)} - 2 g^{(1)\,'} + 3F'' g^{(1)} = -T + g^{(0)\,'}_{(0)} \tag{4.15a}$$

$$\mathcal{L}_1[h^{(1)}_{(0)}] - 2F' h^{(1)}_{(0)} = h^{(0)}_{(0)} - 3 g^{(1)}_{(0)} T' \tag{4.15b}$$

$$\mathcal{L}_1[k_K{}^{(1)}_{(0)}] - 2F' k_K{}^{(1)}_{(0)} = k_K{}^{(0)}_{(0)} - 3 g^{(1)}_{(0)} Y_K' \tag{4.15c}$$

under the boundary conditions

$$g^{(1)\,'}_{(0)}(0) = 0$$

$$h^{(1)}_{(0)}(0) = 0$$

$$k_F{}^{(1)}_{(0)}(0) = y_{Fw}{}^{(1)}_{(0)}$$

$$h^{(1)\,'}_{(0)}(0) = -3 g^{(0)}_{(0)}(0)/B_\infty$$

$$k_F{}^{(1)\,'}_{(0)}(0) = -y_{Fw}{}^{(1)}_{(0)} F(0) + 3(1-Y_{Fw}) g^{(1)}_{(0)}(0)$$

$$g^{(1)\,'}_{(0)}(\infty) = 0$$

$$h^{(1)}_{(0)}(\infty) = 0 \tag{4.16}$$

$$k_O{}^{(1)}_{(0)}(\infty) = 0$$

and the flame conditions, Eq's. (4.2). Again, and it is true for all orders of the velocity expansion, an integral of Eq. (4.15a) is

$$h^{(1)}_{(0)} = \hat{a}_K k_{K\,(0)}^{(1)}$$

and $y_{F_{W(0)}}^{(1)} = 0$ is implied. Therefore, the numerical integration of this problem merely requires an iteration for the two quantities $g^{(1)}_{(0)}(0)$ and $g^{(1)\,''}_{(0)}(0)$. The previously outlined numerical techniques were used and the results are presented in Figures 15 and 16.

Recall that, of necessity from the expansion procedure, any quantity is 90° out of phase with the velocity.[1] It is perhaps best to construct a standing wave at this point and consider components in phase with the pressure. If in Eq. (3.22) $\delta = -1$ and another perturbation with $\delta = 1$ is subtracted there results a standing wave of the form

$$\breve{u} \propto -i \sin(\omega \breve{x} M) e^{i\omega t}$$

and

$$\breve{p} \propto \cos(\omega M \breve{x}) e^{i\omega t}$$

Therefore, for the $\substack{(1)\\(0)}$ problem if a quantity is positive it is in phase with the pressure on the left hand side of the pressure node and vice versa. Speaking, then, of the left side of the pressure node the following observations may be drawn for the fundamental standing mode from

1. This is true for the fundamental frequency only, not for the higher harmonics.

Figure 15:

1. The perturbation in the burning rate is generally in phase with the pressure and higher for higher steady state rates. At sufficiently low $Y_{O\infty}/j$, however, the function becomes negative - out of phase with the pressure.
2. Once again the burning rate perturbation closely follows the flame movement,
3. Again, as the vaporization rate increases the shear stress decreases.
4. The vaporization rate perturbation takes both positive and negative values, being in phase with the pressure for high $Y_{O\infty}/j$ or possibly for sufficiently low Y_{F_W}.

The observations relevant to Figure 16 are

1. Quantities are linear in γ and τ, a result which apparantly can be deduced from the differential equations and boundary conditions.
2. Weakly dependent upon τ, $M_{F_{f(0)}}^{(1)}/\bar{m}_{F_f}$ is in phase with the pressure for sufficiently high γ and again follows the trend of $\tilde{\eta}_{f(0)}^{(1)}$.
3. The vaporization rate perturbation is in phase with the pressure for sufficiently high γ and τ.

4. The shear stress shows the opposite trends as the vaporization rate.

Now effects in phase with the pressure are known to $O[2\xi]$ or first order in frequency. The integration of the low frequency series will not be carried any further. It remains to investigate the asymptotic behavior of high frequency. It is hoped that this information together with information over the full frequency range for the stagnation point will aid in giving full frequency information for the flat plate.

C. The Flat Plate - High Frequency

It was pointed out in Chapter II that when the frequency becomes sufficiently high a phenomenon known as the high frequency boundary layer, a region of rapid transition just as the regular steady state boundary layer, comes into play. For note from Eq's. (3.42), when $\xi \rightarrow \infty$ the dominant terms of the differential equation do not contain the highest derivatives and a formal series development in powers of $1/2\xi$ could not satisfy all boundary conditions. The physics of what is happening has been explained on p. 38. In order to recover the highest derivative it should be clear from Eq's. (3.42) that differentiation by η twice must raise the order of a quantity by 2ξ. Therefore, the scale in which the rapid transitions must take place in the vicinity of boundaries is $1/\sqrt{\omega}$, just as with the plane and spherically symmetric models. The high frequency limit is taken under the constraint that wave propagation normal to the boundary layer is not important. This is an effect of $O\left[\frac{\omega M}{\sqrt{Re}}\right]$ which is the ratio of a normal wave propagation time to a cycle time. This effect is not included as a result of the expansion procedure used in Eq's. (3.14) with the result of Eq. (3.15d).

Therefore, define the high frequency variables to be

$$\alpha = \xi^{-1/2} \qquad \eta = \eta \tag{4.17a}$$

$$\beta_1 = \frac{\sqrt{2}\,\eta}{\alpha} \qquad \beta_2 = \frac{\sqrt{2}}{\alpha}(\eta - \bar{\eta}_f) \tag{4.17b}$$

Eq's. (4.17a) are the outer variables holding for the majority of the field in which diffusion is not sufficiently fast to keep up with time rates of storage of mass, momentum, and energy. Eq's. (4.17b) are the inner variables holding near the surface and flame boundaries where rapid transitions in a distance of 0 $[\alpha]$ must take place. In these two sets of variables Eq's. (3.42) become

$$P_\eta + \delta M[F'P_\eta - F''P] = (1+\delta M)T + \frac{\alpha^2}{2}\left[P_{\eta\eta\eta} + FP_{\eta\eta} + F''P - \gamma\delta MF'''\right] + \frac{\alpha^3}{2}\left[P_{\eta\alpha}F' - P_\alpha F''\right] \tag{4.18a}$$

$$(1+\delta MF')\sigma = \delta MPT' - \delta M(\gamma-1)T + \frac{\alpha^2}{2}\left[\sigma_{\eta\eta} + F\sigma_\eta + PT' - \gamma\delta MT''\right] + \frac{\alpha^3}{2}\left[\sigma_\alpha F' - T'P_\alpha\right] \tag{4.18b}$$

$$(1+\delta MF')y_K = \delta MPY_K' + \frac{\alpha^2}{2}\left[y_{K\eta\eta} + Fy_{K\eta} + PY_K' - \gamma\delta MY_K''\right] + \frac{\alpha^3}{2}\left[y_{K\alpha}F' - Y_K'P_\eta\right] \tag{4.18c}$$

$$P_{\beta\beta\beta} - P_\beta(1+\delta MF') = -\frac{\alpha}{\sqrt{2}}\left[\delta MF'P + (1+\delta M)T - FP_{\beta\beta}\right]$$
$$+ \frac{\alpha^2}{2}\left[\beta P_{\beta\beta} + P_\beta\right]F' + \frac{\alpha^3}{2\sqrt{2}}\Big[\gamma\delta MF''' - PF''$$
$$-\frac{1}{2}\left(F''\beta P_\beta + \sqrt{2}\,F'P_{\beta\alpha}\right)\Big] + \frac{\alpha^4}{2\sqrt{2}}P_\alpha F'' \tag{4.19a}$$

$$\sigma_{\beta\beta} - (1+\delta MF')\sigma = -\delta MPT' + \delta M(\gamma-1)T - \frac{\alpha}{\sqrt{2}}F\sigma_\beta$$
$$-\frac{\alpha^2}{2}\left[PT' - \gamma\delta MT'' - F'\beta\sigma_\beta + \beta T'P_\beta\right]$$
$$-\frac{\alpha^3}{2}\left[F'\sigma_\alpha - T'P_\alpha\right] \tag{4.19b}$$

$$y_{K\beta\beta} - (1+\delta MF')y_K = -\delta MPY_K' - \frac{\alpha}{\sqrt{2}}Fy_{K\beta}$$
$$-\frac{\alpha^2}{2}\left[PY_K' - \gamma\delta MY_K'' - F'\beta y_{K\beta} + \beta Y_K'P_\beta\right]$$
$$-\frac{\alpha^3}{2}\left[F'y_{K\alpha} - Y_K'P_\alpha\right] \tag{4.19c}$$

where Eq's. (4.19) are valid for $\beta = \beta_1$ or $\beta = \beta_2$.

On each side of the flame the existence of a uniformly valid composite expansion of the form

$$P = H(\alpha,\eta) + R_1(\alpha,\beta_1) + R_2(\alpha,\beta_2)$$
$$\sigma = S(\alpha,\eta) + U_1(\alpha,\beta_1) + U_2(\alpha,\beta_2)$$
$$y_K = V_K(\alpha,\eta) + W_{K_1}(\alpha,\beta_1) + W_{K_2}(\alpha,\beta_2) \tag{4.20}$$

is assumed. The solutions with β_1, are valid near the

the surface, those with β_2 near the flame and those with β_1 do not exist on the oxidizer side of the flame.

If Eq's. (4.19) are to be important only near the flame and surface, that is within a distance of order $\beta = 1$ of these boundaries, $F(\eta) = F\left(\frac{\alpha\beta}{\sqrt{2}}\right)$ should possess an expansion

$$F = \sum_{n=0}^{\infty} a_{w_n} (\alpha\beta_1)^n \tag{4.21}$$

near the wall and

$$F = \sum_{n=0}^{\infty} a_{f_n} (\alpha\beta_2)^n \tag{4..22}$$

near the flame. The first terms in α of these expansions are F(0) and F_f, respectively; these are known numbers. Therefore, viewing the homogeneous parts of Eq's.(4.19) it should be clear that solutions can be picked which have exponential decay from the boundaries in the variable β. They can therefore match no power of α of another solution which is $0[\eta \sim 1]$ away from the boundary of interest. Therefore, with these inner solutions, what happens at the flame can have influence on that at the wall for no power of α. This greatly simplifies the boundary conditions. A further expansion which is assumed is

$$H = \sum_{n=0}^{\infty} \alpha^n h^{(n)}(\eta)$$
$$S = \sum_{n=0}^{\infty} \alpha^n s^{(n)}(\eta)$$

$$V_k = \sum_{n=0}^{\infty} \alpha^n v_k^{(n)}(\eta)$$

$$R = \sum_{n=0}^{\infty} \alpha^n r^{(n)}(\beta)$$

$$U = \sum_{n=0}^{\infty} \alpha^n u^{(n)}(\beta)$$

$$W_k = \sum_{n=0}^{\infty} \alpha^n w_k^{(n)}(\beta) \tag{4.22}$$

Then from Eq's. (4.3), (4.20), and (4.22) the boundary conditions at the wall become

No Slip

$$n = 0 \qquad r_{F_i}^{(0)'}(0) = 0$$

$$n \geq 0 \qquad h_F^{(n)'}(0) + \sqrt{2}\, r_{F_i}^{(n+1)'}(0) = 0 \tag{4.23a}$$

Temperature

$n \geq 0$

$$s_F^{(n)}(0) + u_{F_i}^{(n)}(0) = 0 \tag{4.23b}$$

Mass Fraction

$n \geq 0$

$$v_F^{(n)}(0) + w_{F_i}^{(n)}(0) = y_{F_w}^{(n)} \tag{4.23c}$$

Wall Heat Transfer

$$h_F^{(0)}(0) + r_{F_i}^{(0)}(0) = 0$$

$$u_{F_i}^{(0)'}(0)\sqrt{2} = -\frac{2\delta M}{B_\infty}\left[h_F^{(1)}(0) + r_{F_i}^{(1)}(0)\right]$$

$$s_F^{(0)'}(0) + \sqrt{2}\, u_{F_i}^{(1)'}(0) = -\frac{\gamma \delta M F(0)}{B_\infty} - \frac{1}{B_\infty}\Big[h_F^{(0)}(0) + r_{F_i}^{(0)}(0)$$

$$+ 2\delta M\big(h_F^{(2)}(0) + r_{F_i}^{(2)}(0)\big)\Big]$$

$n \geq 1$

$$s^{(n)'}(0) + \sqrt{2}\, u_{F_i}^{(n+1)'}(0) = -\frac{1}{B_\infty}\Big\{2\delta M\big[h_F^{(n+2)}(0) + r_{F_i}^{(n+2)}(0)\big]$$

$$+ (n+1)\big[h_F^{(n)}(0) + r_{F_i}^{(n)}(0)\big]\Big\} \qquad (4.23d)$$

Wall Mass Transfer

$$h_F^{(0)}(0) + r_{F_i}^{(0)}(0) = 0$$

$$\omega_{F_i}^{(0)'}(0)\sqrt{2} = (1 - Y_{F_w})\, 2\delta M\big[h_F^{(1)}(0) + r_{F_i}^{(0)}(0)\big]$$

$$V_F^{(0)'}(0) + \sqrt{2}\, \omega_{F_i}^{(1)'}(0) = \gamma \delta M (1 - Y_{F_w}) - F(0)\big[V_F^{(0)}(0)$$

$$+ \omega_{F_i}^{(0)}(0)\big] + (1 - Y_{F_w})\Big\{h_F^{(0)}(0) + r_{F_i}^{(0)}(0)$$

$$+ 2\delta M\big[h_F^{(2)}(0) + r_{F_i}^{(2)}(0)\big]\Big\}$$

$n \geq 1$

$$V_F^{(n)'}(0) + \sqrt{2}\, \omega_{F_i}^{(n+1)'}(0) = -F(0)\big[V_F^{(n)}(0) + w_{F_i}^{(n)}(0)\big]$$

$$+ (1 - Y_{F_w})\Big\{(n+1)\big[h_F^{(n)}(0) + r_{F_i}^{(n)}(0)\big]$$

$$+ 2\delta M\big[h_F^{(n+2)}(0) + r_{F_i}^{(n+2)}(0)\big]\Big\} \qquad (4.23e)$$

Expanding in the same way at the flame Eq's. (4.2) become

Mass Fraction

$$\dot{j}\left[v_F^{(n)} + w_{F_o}^{(n)}\right] + \left[v_o^{(n)} + w_{o_i}^{(n)}\right] = 0 \tag{4.24a}$$

Temperature

$$s_o^{(n)} - s_F^{(n)} + u_{o_i}^{(n)} - u_{F_o}^{(n)} = q\left[v_F^{(n)} + w_{F_o}^{(n)}\right] \tag{4.24b}$$

Mass Transfer

$$w_{F_o}^{(o)\prime}\dot{j} + w_{o_o}^{(o)\prime} = 0$$

$$n \geq 0, \quad \dot{j}\left[v_F^{(n)\prime} + \sqrt{2}\, w_{F_o}^{(n+1)\prime}\right] + \left[v_o^{(n)\prime} + \sqrt{2}\, w_{o_i}^{(n+1)\prime}\right] = 0 \tag{4.24c}$$

Heat Source

$$u_{o_i}^{(o)\prime} - u_{F_o}^{(o)\prime} = q\, w_{F_o}^{(o)\prime}$$

$$n \geq 0, \quad s_o^{(n)\prime} - s_F^{(n)\prime} + \sqrt{2}\left[u_{o_i}^{(n+1)\prime} - u_{F_o}^{(n+1)\prime}\right] = q\left[v_F^{(n)\prime} + \sqrt{2}\, w_{F_o}^{(n+1)\prime}\right] \tag{4.24d}$$

Continuity

$$r_{F_o}^{(o)\prime\prime} = r_{o_i}^{(o)\prime\prime}$$

$$r_{F_o}^{(1)\prime\prime} = r_{o_i}^{(1)\prime\prime}$$

$$r_{F_o}^{(1)\prime} = r_{o_i}^{(1)\prime}$$

$n \geq 0$

$$h_F^{(n)} + r_{F_o}^{(n)} = h_o^{(n)} + r_{o_i}^{(n)}$$

$$h_F^{(n)\prime} + \sqrt{2}\, r_{F_o}^{(n+1)\prime} = h_o^{(n)\prime} + \sqrt{2}\, r_{o_i}^{(n+1)\prime} \tag{4.24e}$$

$$h_F^{(n)\prime\prime} + 2\, r_{F_o}^{(n+2)\prime\prime} = h_o^{(n)\prime\prime} + 2\, r_{o_i}^{(n+2)\prime\prime}$$

At infinity Eq's. (4.4) become

$$h_0^{(0)\prime}(\infty) = 1 \qquad n \geq 1 \quad h_0^{(n)\prime}(\infty) = 0 \tag{4.25a}$$

$$s_0^{(0)}(\infty) = -\delta M(\gamma-1) \qquad n \geq 1 \quad s^{(n)}(\infty) = 0 \tag{4.25b}$$

$$V_0^{(n)}(\infty) = 0 \tag{4.25c}$$

where the errors made are $0[e^{-1/\alpha}]$ in all the boundary conditions if, indeed, the inner solutions demonstrate exponential decay to all orders in α.

Eq's. (4.18), (4.19), and (4.22) become

$$(1+\delta M)h^{(0)\prime} - \delta M F'' h^{(0)} = (1+\delta M)T \tag{4.26a}$$

$$(1+\delta M)h^{(1)\prime} - \delta M F'' h^{(1)} = 0 \tag{4.26b}$$

$$(1+\delta M)h^{(2)\prime} - \delta M F'' h^{(2)} = \tfrac{1}{2}[h^{(0)\prime\prime\prime} + F h^{(0)\prime\prime} + F'' h^{(0)} - \gamma\delta M F'''] \tag{4.26c}$$

$n \geq 3$

$$(1+\delta M)h^{(n)\prime} - \delta M F'' h^{(n)} = \tfrac{1}{2}[h^{(n-2)\prime\prime\prime} + F h^{(n-2)\prime\prime} + F'' h^{(n-2)}] + \left(\frac{n-2}{2}\right)[h^{(n-2)\prime} F' - F'' h^{(n-2)}] \tag{4.26d}$$

$$(1+\delta M)s^{(0)} = \delta M h^{(0)} T' - \delta M(\gamma-1)T \tag{4.27a}$$

$$(1+\delta M)s^{(1)} = \delta M h^{(1)} T' \tag{4.27b}$$

$$(1+\delta M)s^{(2)} = \tfrac{1}{2}[s^{(0)\prime\prime} + F s^{(0)\prime} + h^{(0)} T' - \gamma\delta M T''] + \delta M h^{(2)} T' \tag{4.27c}$$

$n \geq 3$

$$(1+\delta M)s^{(n)} = \delta M h^{(n)} T' + \tfrac{1}{2}[s^{(n-2)\prime\prime} + F s^{(n-2)\prime} + h^{(n-2)} T'] + \tfrac{1}{2}(n-2)[F' s^{(n-2)} - T' h^{(n-2)}] \tag{4.27d}$$

$$(1+\delta M)V_K^{(0)} = \delta M h^{(0)} Y_K' \tag{4.28a}$$

$$(1+\delta M)\, V_K^{(1)} = \delta M\, h^{(1)}\, Y_K' \tag{4.28b}$$

$$(1+\delta M)\, V_K^{(2)} = \delta M\, h^{(2)}\, Y_K' + \tfrac{1}{2}\left[V_K^{(0)''} + F V_K^{(0)'} + h^{(0)} Y_K' + \gamma \delta M Y_K''\right] \tag{4.28c}$$

$n \geq 3$

$$\begin{aligned}(1+\delta M)\, V_K^{(n)} = \delta M\, h^{(n)}\, Y_K' &+ \tfrac{1}{2}\left[V_K^{(n-2)''} + F V_K^{(n-2)'} + h^{(n-2)} Y_K'\right] \\ &+ \left(\tfrac{n-2}{2}\right)\left[F' V_K^{(n-2)} - F'' h^{(n-2)}\right]\end{aligned} \tag{4.28d}$$

$$\frac{d}{d\beta}\mathcal{L}_4\left[r_{F_i}^{(n)}\right] = \frac{d}{d\beta}\left[\frac{d^2}{d\beta^2} - 1\right] r_{F_i}^{(n)} = 0 \tag{4.29a}$$

$$\mathcal{L}_4\left[u_{F_i}^{(n)}\right] = \left(\frac{d^2}{d\beta^2} - 1\right) u_{F_i}^{(n)} = 0 \tag{4.29b}$$

$$\mathcal{L}_4\left[w_{F_i}^{(n)}\right] = 0 \tag{4.29c}$$

$$\frac{d}{d\beta}\mathcal{L}_5\left[r_{F_0}^{(n)}\right] = \frac{d}{d\beta}\left[\frac{d^2}{d\beta^2} - (1+\delta M F_f')\right] r_{F_0}^{(n)} = 0 \tag{4.30a}$$

$$\mathcal{L}_5\left[u_{F_0}^{(n)}\right] = 0 \tag{4.30b}$$

$$\mathcal{L}_5\left[w_{F_0}^{(n)}\right] = 0 \tag{4.30c}$$

$$\frac{d}{d\beta}\mathcal{L}_5\left[r_{o_i}^{(n)}\right] = 0 \tag{4.31a}$$

$$\mathcal{L}_5\left[u_{o_i}^{(n)}\right] = 0 \tag{4.31b}$$

$$\mathcal{L}_5\left[w_{o_i}^{(n)}\right] = 0 \tag{4.31c}$$

In Eq's. (4.29)-(4.31) only the leading order differential equations in α , where n is the order of the first non-zero solution, have been given. The higher orders will only add inhomogeneous parts. In addition to the nomenclature list it will be pointed out here that the first subscript refers to which side of the flame is under consideration and the second subscript refers to either the wall (i) or flame (o) or the flame (i) on the oxidizer side. The method of separation of terms in Eq's. (4.26)-(4.31) has been possible due to the linearity of the problem; quantities due to time rates of storage have been incorporated into the H , S , and V_K equations. This system under the boundary conditions, Eq's. (4.23)-(4.25), can now be easily recursively solved. The procedure is as follows:

1. Eq's. (4.26) may all be solved by one integration yielding one constant of integration each time , $A_{h_{F,O}}^{(n)}$, which can, however, undergo a jump across the flame .
2. Eq's. (4.27) and (4.28) directly determine $\phi_{F,O}^{(n)}$ and $v_{F,O}^{(n)}$ in terms of $A_{h_{F,O}}^{(n)}$.
3. Eq's. (4.29)-(4.31) can all be solved, each equation yielding two contants of integration. One constant is taken as zero to provide exponential decay into the region of interest.

Doing this

$$h^{(0)} = A_{h^{(0)}} e^{\delta M \int_0^\eta \frac{F'' d\eta}{1+\delta M F'}} + (1+\delta M)\, e^{\delta M \int_0^\eta \frac{F'' d\eta}{1+\delta M F'}} \int_0^\eta e^{-\delta M \int_0^\eta \frac{F'' d\eta}{1+\delta M F'}}\, T\, d\eta$$

$$h^{(1)} = A_{h^{(1)}} e^{\delta M \int_0^\eta \frac{F'' d\eta}{1+\delta M F'}}$$

etc.

(4.32)

Therefore $s^{(0)}$, $s^{(1)}$, $v^{(0)}$, $v^{(1)}$, etc. are determined. Consider Eq's. (4.29)

$$r_{F_i}^{(0)} = A_{r_{F_i}^{(0)}} e^{-\beta_1}$$

$$u_{F_i}^{(0)} = A_{u_{F_i}^{(0)}} e^{-\beta_1}$$

$$w_{F_i}^{(0)} = A_{w_{F_i}^{(0)}} e^{-\beta_1}$$

(4.33)

Now consider the wall problem. Eq's. (4.23a) say $A_{r_{F_i}^{(0)}} = 0$. Therefore, Eq's. (4.23d) say $A_{h_F^{(0)}} = 0$. Since $r_{F_i}^{(0)} = 0$, the same differential equation in Eq's. (4.29) must hold for $r_{F_i}^{(1)}$. This will always happen if the leading order solution is zero. Applying Eq. (4.23b) to Eq's. (4.29b) and (4.27a)

$$A_{u_{F_i}^{(0)}} = \delta M(\gamma-1)\tau$$

Now apply the no slip condition, Eq's. (4.23a). Since $r_{F_i}^{(1)} = A_{r_{F_i}^{(1)}} e^{-\beta_1}$,

$$A_{r_{F_i}^{(1)}} = -\frac{(1+\delta M)\tau}{\sqrt{2}}$$

Then apply the heat transfer condition, Eq. (4.23d), in which the only unknown is $A_{h^{(1)}}$, and

$$A_{h_F^{(1)}} = \frac{2}{\sqrt{2}}\left[B_\infty(\gamma-1)-1-\delta M\right]$$

Then from Eq. (4.33) apply the mass transfer condition, Eq's. (4.23c), to obtain

$$A_{w_{F_i}^{(0)}} = -\delta M(1-Y_{F_w})B_\infty(\gamma-1)\tau$$

Then from Eq's. (4.23c) and (4.28a) $y_{F_w}^{(0)}$ is determined as

$$y_{F_w}^{(0)} = A_{w_{F_i}^{(0)}}$$

The recursive procedure is then as follows:

1. Eq's. (4.23a) determine $A_{r_{F_i}^{(n+1)}}$.
2. Eq's. (4.23b) determine $A_{h_F^{(n+1)}}$.
3. Eq's. (4.23d) determine $A_{u_{F_i}^{(n+1)}}$.
4. Eq's. (4.23e) determine $A_{w_{F_i}^{(n)}}$.
5. Eq's. (4.23c) determine $y_{F_w}^{(n)}$.

The wall problem is therefore completely solved without reference to the flame and all $A_{h_F^{(n)}}$ are determined. From Eq. (4.3d) the wall mass vaporization rate perturbation has the expansion

$$M_w\sqrt{2x} = -\left[\left(1+\frac{2\delta M}{\alpha^2}\right)P+\alpha P_\alpha\right]_w = \sqrt{2x}\sum_{-1}^{\infty}\alpha^n M_w^{(n)}$$

and

$$\frac{M_w^{(-1)}}{\overline{m}_w} = -\frac{2\delta M}{F(0)}\left[h^{(1)}(0) + r_{F_i}^{(1)}(0)\right] = \frac{\sqrt{2}\,\delta M B_a \tau(\gamma-1)}{F(0)} \tag{4.34}$$

a result which should obviously be compared with Eq's. (2.7) and (2.8) for the plane and spherically symmetric models. The vaporization rate goes to infinity as $\sqrt{i\omega\chi}$ with a component in phase with the pressure and is purely a pressure effect.

Now consider the problem at infinity. Eq's. (4.26)-(4.28) automatically satisfy Eq's. (4.25). In fact that is what partly dictated the form of the separation of the differential equations. Then only the flame problem remains. The procedure for solution is as follows:

1. All $h_o^{(n)}$ are known but the constants of integration on the oxidizer side, $A_{h_o}^{(n)}$, are unknown.
2. $\vartheta_o^{(n)}$ and $v_o^{(n)}$ are known in terms of $A_{h_o}^{(n)}$.
3. Eq's. (4.30) and (4.31) can all be solved, always picking the solution with exponential decay into the region of interest.

Then Eq's. (4.24e) say that

$$r_{F_o}^{(0)} = r_{F_o}^{(1)} = r_{O_i}^{(0)} = r_{O_i}^{(1)} = 0$$

and that $h^{(0)}$ and $h^{(1)}$ are continous across the flame. Then Eq's. (4.27a and b), (4.28a and b) under Eq's. (4.24a-d) say that

$$W_{F_0}^{(0)} = W_{F_0}^{(1)} = W_{o_i}^{(0)} = W_{o_i}^{(1)} = u_{o_i}^{(0)} = u_{o_i}^{(1)} = u_{F_0}^{(0)} = u_{F_0}^{(1)} = 0$$

This is already striking and is analogous to a result obtained in Appendix C for the spherical model. The field under high frequencies of oscillation responds naturally to this oscillation at the flame. Boundary conditions are natural to this oscillation so that strong gradients do not appear at the flame. Therefore, it will be expected that the mass burning rate perturbation will remain bounded in the limit of infinite frequency.

Continuing, from Eq's. (4.30a) and (4.31a)

$$r_{F_0}^{(2)} = A_{r_{F_0}^{(2)}} e^{\beta_2 \sqrt{1+\delta M F_f'}}$$

$$r_{o_i}^{(2)} = A_{r_{o_i}^{(2)}} e^{-\beta_2 \sqrt{1+\delta M F_f'}}$$

with analogous solutions to the remaining equations in Eq's. (4.30) and (4.31). Since from Eq's. (4.26b) $h^{(1)\prime}$ is continous, Eq's. (4.24e) say

$$A_{r_{F_0}^{(2)}} = -A_{r_{o_i}^{(2)}}$$

Then again Eq's. (4.24e) say

$$h_F^{(0)\prime\prime} - h_o^{(0)\prime\prime} = 4(1+\delta M F_f')\, A_{r_{o_i}^{(2)}}$$

Eq. (4.26a) then yields

$$A_{r_{o_i}^{(2)}} = \frac{a_F - a_o}{4} F_f'' \frac{(1+\delta M)}{(1+\delta M F_f')^2}$$

Then the fourth of Eq's. (4.24e) gives a jump condition on $A_{\ell(2)}$. This exact procedure is then repeated starting with $r_{F_0}^{(3)}$ and $r_{O_i}^{(3)}$ and the entire solution to the stream function is generated. The constants of integration in Eq's. (4.30b and c) and (4.31b and c) are obtained in the following manner from Eq's. (4.27), (4.28), and (4.24):

1. Eq's. (4.24a and c) are two equations in the two unknowns $A_{w_{F_0}}^{(n)}$ and $A_{w_{O_i}}^{(n)}$.
2. Eq's. (4.24b and d) are the two equations for $A_{u_{F_0}}^{(n)}$ and $A_{u_{O_i}}^{(n)}$.

Carrying this out part way

$$A_{w_{O_i}^{(2)}} = -j\, A_{w_{F_0}^{(2)}} = \delta M j (c_F F_f'')^2 g$$

from which the burning rate at the flame can be carried out to $0[\alpha]$. Expanding Eq. (4.13) in the present variables,

$$\sqrt{2x}\, M_{F_f} = \sqrt{2x} \sum_0^\infty \alpha^n M_{F_f}^{(n)}$$

The outlined procedure yields

$$\frac{M_{F_f}}{\bar{m}_{F_f}} = -\delta M\left[\gamma - T_f + \bar{m}_{F_f}\, g\, \alpha + O[\alpha^2]\right] \tag{4.35}$$

As $\alpha \to 0$ the physical origin of the terms can be traced through the equations. The γ factor comes from a compression of the boundary layer increasing gradients and transport properties. T_f comes from a convective "sweeping" term carrying the steady state fuel flow into the flame and by moving the flame position.

T_f appears because the convective sweeping is increased when the pressure gradient acts on a fluid of lower density at the flame.[1] Note that Eq. (4.35) is finite as $\alpha \to 0$ and is purely a pressure effect to $0[\alpha]$. It approaches the infinite frequency limit from a side which has a component in phase with the pressure, although at $\alpha = 0$ since $T_f > \gamma$ usually, it is generally 180° out of phase with the pressure.

Because of the algebraic complexity of the high frequency expansion, it has not been carried further. It will be returned to, however, after completion of the stagnation point analysis. Many of the results are similar.

D. The Stagnation Point

Eq's. (3.60) are to be solved under the wall boundary conditions, obtained from Eq's. (3.16),

$$P'(0) = 0$$

$$\sigma(0) = 0$$

$$y_F(0) = y_{F_w}$$

$$y_F'(0) = \frac{4}{3}(1 - Y_{F_w})[P(0) + \gamma \delta M F(0)] - \frac{4}{3} y_{F_w} F(0)$$

$$\sigma'(0) = \frac{4}{3 B_\infty}\left[-P(0) - \gamma \delta M F(0)\right] \tag{4.36}$$

1. This is a consequence of Eq's. (4.26a) and (4.28a).

the flame conditions, Eq's. (4.2), with Eq. (4.2e) replaced by the statement

$$P, P', P'' \quad \text{continuous}$$

and the conditions at infinity, Eq's. (4.4). Once again it is desirable to split the problem into a velocity effect and a pressure effect by assuming

$$\begin{aligned} P &= P_{(0)} + \delta M P_{(1)} \\ y_K &= y_{K_{(0)}} + \delta M y_{K_{(1)}} \\ \sigma &= \sigma_{(0)} + \delta M \sigma_{(1)} \end{aligned} \tag{4.37}$$

Using Eq's. (4.37) in Eq's. (3.60) the problems are separated. Using the linear operators

$$\mathcal{L}_2 = \frac{d^3}{d\eta^3} + \frac{4}{3} F \frac{d^2}{d\eta^2} - \frac{4}{3} F' \frac{d}{d\eta} + \frac{4}{3} F'' - \frac{4}{9} i\omega \frac{d}{d\eta}$$

$$\mathcal{L}_3 = \frac{d^2}{d\eta^2} + \frac{4}{3} F \frac{d}{d\eta} - \frac{4}{9} i\omega$$

the equations become

$$\begin{aligned} \mathcal{L}_2[P_{(0)}] &= -\frac{4}{9}\left[T(3+i\omega) + \frac{3}{2}\sigma_{(0)}\right] \\ \mathcal{L}_3[\sigma_{(0)}] &= -\frac{4}{3} P_{(0)} T' \\ \mathcal{L}_3[y_{K_{(0)}}] &= -\frac{4}{3} P_{(0)} Y_K' \end{aligned} \tag{4.38}$$

$$\begin{aligned} \mathcal{L}_2[P_{(1)}] &= \gamma F''' - \frac{2}{3} T(\gamma - 1) - \frac{2}{3}\sigma_{(1)} - \frac{40}{27} T i\omega\left(\frac{i\omega}{3} + 1\right) \\ \mathcal{L}_3[\sigma_{(1)}] &= -\frac{4}{3} P_{(1)} T' + \frac{4}{9} i\omega(\gamma - 1) T + \gamma T'' \\ \mathcal{L}_3[y_{K_{(1)}}] &= -\frac{4}{3} P_{(1)} Y_K' + \gamma Y_K'' \end{aligned} \tag{4.39}$$

Eq's. (4.36) become

$$P'_{(0)}(0) = 0$$

$$\sigma_{(0)}(0) = 0$$

$$y_{F_{(0)}}(0) = y_{F_{W_{(0)}}}$$

$$y'_{F_{(0)}}(0) = \frac{4}{3}(1 - Y_{F_W})\,P_{(0)}(0) - \frac{4}{3}\,y_{F_{W_{(0)}}}F(0)$$

$$\sigma'_{(0)}(0) = -\frac{4}{3}\,\frac{P_{(0)}(0)}{B_\infty} \tag{4.40}$$

$$P'_{(1)}(0) = 0$$

$$\sigma_{(1)}(0) = 0$$

$$y_{F_{(1)}}(0) = y_{F_{W_{(1)}}}$$

$$y'_{F_{(1)}}(0) = \frac{4}{3}(1 - Y_{F_W})[\gamma F(0) + P_{(1)}(0)] - \frac{4}{3}F(0)\,y_{F_{W_{(1)}}} \tag{4.41}$$

$$\sigma'_{(1)}(0) = -\frac{4}{3B_\infty}[\gamma F(0) + P_{(1)}(0)]$$

The flame conditions remain unchanged in form and Eq's. (4.4) become

$$P'_{(0)}(\infty) = 1$$

$$\sigma_{(0)}(\infty) = 0$$

$$y_{O_{(0)}}(\infty) = 0 \tag{4.42}$$

$$P_{(1)}'(\infty) = \frac{10}{9} i\omega$$

$$\sigma_{(1)}(\infty) = 1-\gamma \qquad (4.43)$$

$$y_{o_{(1)}}(\infty) = 0$$

Eq's. (4.38), (4.40), and (4.46) comprise the velocity problem which has the integral

$$\sigma_{(0)} = \hat{a}_K \, y_{K_{(0)}}$$

specifying $y_{F_{W_{(0)}}} = 0$. Once again $y_o/\dot{\gamma}$ is the actual variable of interest and $\dot{\gamma}$ does not have to be independently specified.

Previously mentioned numerical integration techniques have been applied to these two problems. The pressure problem requires a sixfold iteration in the real and imaginary parts of $P(0)$, $P''(0)$, and y_{F_W} . Only a four-fold iteration is required for the velocity problem because of the mass fraction integral. To check the numerical procedure, however, a high frequency analysis has also been performed for the stagnation point. Here the appropriate high frequency variables are

$$\alpha = \frac{1}{\sqrt{i\omega}} \qquad \beta_1 = \frac{\eta}{\alpha} \qquad \beta_2 = \frac{\eta - \bar{\eta}_f}{\alpha} \qquad (4.44)$$

Here $\alpha \neq \alpha(x)$, however. The same expansion techniques as in Eq's. (4.20)-(4.22) are used. The methodology follows much the same pattern as for the flat plate; some differences must, however, appear because the stagnation point feels different free stream gradients than the flat plate.

Since the procedures are so similar only the primary results will be presented here. The analysis is located in Appendix E. Note that the high frequency analysis is carried out without making the formal split into a pressure and velocity problem.

The perturbation in surface mass fraction becomes

$$y_{F_w} = -\delta M(1-Y_{F_w})(\gamma-1)\, B_\infty \tau \; + O[\alpha] \tag{4.45}$$

which is finite and should be compared to the result on p. 118 for the flat plate. Analogous to Eq. (4.34) there is obtained

$$\frac{M_w}{\bar{m}_w} = -\frac{\delta M(\gamma-1)\,\tau\, B_\infty}{2\,F(0)\,\alpha} \; + O[1] \tag{4.46}$$

and the same conclusions are drawn as were for the flat plate. The burning rate perturbation becomes

$$\frac{M_{F_f}}{\bar{m}_{F_f}} = \delta M\left(\frac{10}{3}T_f - \gamma\right) - \alpha\,\delta M\, 5\,\bar{m}_{F_f}\, q + \alpha^2\, 3T_f + O[\delta M \alpha^2] + O[\alpha^3] \tag{4.47}$$

which is identical in form with Eq. (4.35). Except for numerical factors the behavior of the flat plate and the stagnation point is the same at high frequency. This is because the wave scattering produces a travelling wave in the same direction as the free stream travelling wave for the flat plate. Only the numerical effect is changed.

It is therefore suspected that a knowledge of the stagnation point behavior over the full frequency range will give good qualitative information for the flat plate over the full frequency range. Indeed, it may even be true over the entire leading edge of the body.

Figures 17-22 contain the results for the pressure effect over a wide frequency range. Consider first Figures 17a and b as representative of typical curves. The following observations are apparent:

1. The quasi-steady response ($\omega = 0$) for $P_r(0)$ and M_{F_f}/m_{F_f} is generally substantially lower than for the flat plate, c.f. Figure 12.[1] However, due to interaction with the convective field this response is greater than for the plane model, c.f. Figure 2.
2. The real part of the mass burning rate perturbation is a monotonically increasing function of frequency. For $\frac{20}{9} T_f > \gamma$ the curve will always cross the zero line and move out of phase with the pressure in the mid-frequency range, remaining so far any further frequency increase. As with the flat plate, the low frequency limit has the burning rate in phase with the pressure.

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1. Again, note the significance of P from Eq. (3.55).

3. The shape of the $P_r(0)$ curve is quite different from that of the plane model, c.f. Figure 2. It is therefore clear that the inclusion of convection and flame movement radically changes the unsteady behavior over the simplified plane model in the mid-frequency range. However, the high frequency limits must be qualitatively the same since the physics are the same. The flame cannot influence the wall and a compression process is controlling, not convection.

4. The skin friction components ($P_r''(0)$ and $P_i''(0)$ in the figures) are monotonic functions of frequency with extremely strong response in the high frequency range. This is, of course, due to a second derivative being taken across the high frequency boundary layer. It should be noted that the skin friction is essentially imaginary, out of phase with the pressure, during a large portion of the frequency range.

5. The surface mass fraction perturbations are extremely low once again justifying the neglect of surface temperature perturbations. From now on this will be considered an uninteresting quantity

and it will not be plotted.

6. As noted before, the flame movement strongly influences the burning rate perturbation as is evidenced by similar shapes of the curves.[1]
7. Since the shear stress and vaporization rate curves are not similar in shape it is clear that the unsteady state modifies previous conceptions concerning the effect of blowing on the shear stress.

A further observation which deserves special mention is that real quantities behave quasi-steadily far out in frequency. Imaginary quantities are nearly linear in frequency all the way out to $\omega \sim 1$. Such behavior is generally not to be expected since as soon as ω^2 is not negligible compared to 1, curvature effects are to be expected. For the plane model unsteady effects become prominent very low in frequency. This strange behavior for the boundary layer has strong implications concerning the practical application of the theory. Recall that on p. 71 it was shown that ω was essentially the ratio of a diffusion time to a cycle time. More precisely, the actual nominal boundary layer thickness, while $0\left[a^*/\sqrt{Re}\right]$, is

$$\delta^* \sim 3-5 \ a^*/\sqrt{Re}$$

1. Note that "flame movement" must also be interpreted as containing a compression component. See Eq. (4.1).

such that

$$\omega \sim \frac{\omega^* t^*_{dif}}{9-25}$$

Therefore, for $\omega^* t^*_{dif}$ 0 [1], $\omega \sim 1/10$. $\omega^* t^*_{dif}$ was the frequency considered in the plane model. It is clear that because of the mathematics of the convective layer the frequency naturally under consideration is less than the previous intuitive interpretation. Moreover, since significant response does not occur until $\omega \sim 1$, it is clear that quite short cycle times compared to diffusion times are implied for significant unsteady response.

Figures 18-22 show the effect of parameter changes for magnitudes of changes which are practically possible in usual systems. If the change produced negligible shift of the curves from the base curve of Figure 17 it was omitted. The following conclusions can be drawn

1. The effect of Y_{F_w} is as it was for the flat plate. Strong effects take place at the surface, but only mild effects at the flame. The directions of changes in the quasi-steady limit are the same for both cases.
2. Y_{O_∞}/j produces the strongest effect of any parameter change. This may have been

guessed because of the influence it has on T_f which strongly influences the high frequency limit. The most important effects appear to be the shifting of the zero point of $P_r(0)$ and M_{F_f}/m_{F_f} to a higher frequency and the production of a minimum point in the burning rate for sufficiently high $Y_{O\infty}/j$. The quasi-steady limits show different behavior between the stagnation point and flat plate. The burning rate responds similarly for changes in $Y_{O\infty}/j$ but the shear stress and vaporization rate behave differently in the two cases.

3. The effects of τ are only felt for the vaporization rate and shear stress for sufficiently high frequency. This is expected since it strongly affects the wall behavior in the high frequency limit. Also, as expected, τ has little effect at the flame and in the quasi-steady limit.
4. g has the same type of effect as $Y_{O\infty}/j$ since it roughly affects the same thing, the flame temperature. However, in the quasi-steady limit the behavior with this parameter is the same as for the flat plate.
5. γ changes make reasonably strong changes

in the vaporization rate behavior as is expected from the high frequency behavior. The burning rate is not significantly affected for the parameters chosen here; however, for sufficiently low T_f it must affect the high frequency behavior in a strong manner. In this respect the flat plate will be more strongly influenced than the stagnation point.

Now turn to the velocity effect. The numerical results are presented in Figures 23-29. Considering Figures 23a and b as representative the following observations are apparent:

1. Again, as with the pressure effect, the real parts of all quantities (parts in phase with the velocity) remain at essentially the quasi-steady values all the way to ω of 0 [1]. The high frequency behavior has essentially been reached when $\omega \sim 20$. The imaginary parts are nearly linear in ω out as far as $\omega \sim 1$.
2. The quasi-steady perturbations are merely due to a perturbation in the Reynolds number through the free stream velocity perturbation. Thus $M_{F_{f(o)}} / \bar{m}_{F_f} \rightarrow 1/2$, etc. (See Eq's. (4.14) for the flat plate).
3. Changes in the shear stress are extremely small until the high frequency effects set

in.

4. Again notice the strong influence of $\tilde{\eta}_f$ on $M_{F_f}/\bar{m}_{F_f}$.

5. The imaginary part of the burning rate perturbation is negative for this case over the full frequency range. Therefore, for a standing wave in the sense discussed on p. 105 for the flat plate it is out of phase with the pressure by 180°. The same is true for the vaporization rate perturbation. Note, however, from the flat plate results that for sufficiently high $Y_{o\infty}/j$, high q or low τ this result may reverse itself.

It should be remembered at this point that the velocity effect can be extremely strong because of the raising in order of magnitude of 1/M compared to the pressure effect. Therefore, if a velocity effect goes in phase with the pressure an important augmentation of the pressure effect will occur. A caution at this point is to note that if $\tilde{\eta}_f$, $P'(0)$, and $P(0)$ were normalized by the steady state quantities all the curves would collapse to a single curve in the quasi-steady limit. Viewing Figure 24-27, the following conclusions can be drawn:

1. The effect of Y_{F_W} is somewhat similar to what would have been predicted by flat plate results. $P_i(0)$ increases, $P_i''(0)$ decreases, and $M_{F_f}/\bar{m}_{F_f}$ is little changed with an increase in Y_{F_W}. The normalized curves of $P_r(0)$ and $P_r''(0)$ would show little change just as with the burning rate perturbation.

2. As expected from the flat plate Y_{O_∞}/j produces strong effects. First, an increase generally shifts the usual behavior to a higher frequency. A peak develops in $M_{F_f}/\bar{m}_{F_f})_r$ $M_{F_f}/\bar{m}_{F_f})_i$ goes positive in the low frequency range. $P_i(0)$ moves toward the negative side but develops a stronger peak on the positive side. Normalized by the steady state value, however, $P''(0)$ shows no radical change.

3. The effect of q_0 is not as strong as might have been believed from the flat plate results. The main effect of an increase in q_0 is to shift the curves to a higher frequency. In the low frequency range, however, the same general trends as for the flat plate are observed.

4. τ has almost no effect except at high frequency for $P''(0)$. An increase in just shifts the curves to a slightly higher frequency.

Since the effects of Y_{o_∞}/j and q were so important for the flat plate in determining where the imaginary parts of the burning and vaporization rates went in phase with the pressure it is desirable to see more closely where this happens for the stagnation point. Figures 28 and 29 contain these results. They are plotted for $\omega=1$ and because of observation 1. on p. 129 they should closely correspond to the first order in frequency flat plate results of Figures 15 and 16. Here, however, the dependences are much milder. It can be seen that for sufficiently high Y_{o_∞}/j and q $M_{F_f}/\bar{m}_{F_f})_i$ can be driven in phase with the pressure for an appropriately constructed standing wave. It is known, however, from the asymptotic behavior of this quantity that it must change sign at a rather low frequency if it is positive for a while. Observe that $P_i(0)$ cannot practically be made to go negative. Therefore, this is a result contrary to those for the flat plate.

E. Feedback in Combustion Systems

It should be apparent by now to the reader that prime emphasis in the results has been given to components of the burning and vaporization rates which can be in phase with the pressure for either a travelling or standing wave. The reason for this has been previously indicated; these components determine, in many practical combustion systems, whether or not a periodic wave can even exist in the system. It is now well established mostly from theoretical work concerning rocket combustors (29, 30, 31) that energy or mass release perturbations per unit volume under the action of an acoustic wave must be in phase with the pressure and of sufficient magnitude to overcome certain damping and sustain such a periodic wave. This is saying that these perturbations must increase the rates per unit mass of gas locally in the chamber. Damping is usually provided mainly by convection of the mean flow out the chamber and by boundary conditions such as a deLaval nozzle.

It should be clear that this is an intrinsic feedback problem between the main combustion gases and the mass-energy sources (droplets in the present work), although other factors may enter such as interaction with the injection system. However, what has been studied here is a forced oscillation, not feedback, of the droplet burning process, and confusion can arise. The reason for this is found in many types of perturbation problems. If flow velocities inside the combustor are sufficiently small so that Mach numbers are small, the wave equation applies to the chamber gases under small perturbations. There is no damping here if M=0 and periodic solutions can be found which are neither damped nor amplified in time (the acoustic modes). Such solutions are correct up to terms of $0[M]$ compared to $0[1]$. Now since the droplet burning process is what is creating the terms of $0[M]$ (The mass-energy source to create the mean flow), it will also contribute terms of this order under perturbation unless an extreme resonance takes place. Thus, a wave of $0[1]$ containing terms of $0[M]$ acts upon the burning process of $0[M]$ to create a feed-back in the term of $0[M]$ of the wave. Therefore, the burning process appears forced by a wave of $0[1]$ and the real feedback takes place in $0[M]$ But this is where the damping occurs so that it is clear that an intrinsic feedback system is under consideration. This argument is, of course, rough but has all been put on rigorous theoretical ground and is common in oscillation theory with feedback occuring through a small parameter.

It is not yet established whether vaporization rate or burning rate perturbations or both are the most important quantities concerning this feedback to the wave. Arguments can be given concerning the importance of both. As such both will be concentrated upon. The important questions are 1.) in what frequency ranges does a strong amplification of these perturbation rates take place and 2.) what is the magnitude of this amplification?

Consider then by way of practical application a simple standing wave in a chamber as constructed on p. 105. Now, however, assume that many harmonics of the fundamental can exist. In particular it is also possible by a Fourier expansion to consider a "standing" shock wave pattern as studied by Sirignano (32) in relation to rocket instability. Each harmonic then has the form

$$\breve{u} \propto \sin(n\omega M \breve{x})\, e^{i[n\omega t - \frac{\pi}{2n}]}$$

$$\breve{p} \propto \cos(n\omega M \breve{x})\, e^{i n\omega t} \qquad n = 1, 2, 3, \cdots$$

where ωM is the fundamental frequency ($\omega M = \pi + O[M]$) and $\breve{x}$ is the chamber variable measured longitudinally from one boundary. Consider the droplets to be located at an $\breve{x}$ such that $\sin(n\pi\breve{x})$ and $\cos(n\pi\breve{x})$ have the same sign, i.e., always on the left hand side of the first pressure node for the harmonic under consideration. It should be clear that all quantities of interest, $M_{F_f}/\overline{m}_{F_f}$, $P(o)$, etc. take on the same form as the velocity for a velocity effect and the same form as the pressure for pressure effect;[1] the proportionality sign is replaced by an equal sign if the Fourier coefficient

1. That is, they are proportional to the right hand sides of the proportionality relations above, respectively.

of the wave type and the appropriate magnitude of the result under consideration are multiplied to the right hand side. In addition the pressure effect must be multiplied by M if the same non-dimensionalization scheme is used as used previously. By way of example consider spherical droplets with $a^* = 500\mu$ and $u_\infty^* = 50$ ft/sec. Then the characteristic diffusion time is $a^*/u_\infty^* = 3.29 \times 10^{-5}$ sec.[1]. This corresponds to a Reynolds number of 67 if $\mu_\infty^* = 10^{-3}$ poise and $\rho_\infty^* = .05$ lb/ft^3. For a sound speed of 3500 ft/sec the Mach number is .014.[2]

The wave dynamic problem is, however, usually solved with respect to a different non-dimensionalization scheme. Velocities are refered to the speed of sound and pressures by the ambient pressure. To put this treatment in this scheme the velocity perturbation becomes divided by M and the pressure by 1/ M. Now the pressure effect appears as $0[1]$ and the velocity effect as $0[1/M]$. Then if $n\pi\gamma$ is $0[M]$ or greater the velocity effect is as strong or stronger than the pressure effect. This point should be strongly emphasized: even though the droplets may be near a velocity node the effect may be extremely strong since

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1. This is of the order of the characteristic times of some high frequency oscillations in liquid propellant rocket chambers.
2. This is a relative velocity Mach number. Actual flow Mach numbers are generally substantially larger in actual combustors.

velocity is raised by an order of magnitude in 1/M in the unsteady state.

Note then that for the wave construction only the real parts for the pressure effects are in phase with the pressure for the fundamental and any harmonic. However, the real parts for the velocity effect, while 90° out of phase with the pressure for the fundamental, have components in phase for the higher harmonics and are directly in phase for $n = \infty$. Simultaneously, the imaginary parts move from in phase with the pressure to 90° out as n goes from 1 to ∞. Discussion of the results in relation to this wave construction are deferred until Chapter V.

F. Summary

Perturbing about the steady state solutions developed in Chapter III, periodic solutions have been obtained to the stagnation point and flat plate approximations to the droplet burning problem. The stagnation point problem yielded solutions over the entire frequency spectrum while the flat plate lent itself only to low and high frequency analysis. Certain correspondences in the two cases have been pointed out. Attention has been drawn to the vaporization and burning rate perturbations as being of prime importance to the problem of combustion feedback to an acoustic wave.

CHAPTER V

REVIEW AND CONCLUSIONS

It has been recognized that the process of droplet burning and vaporization in an actual combustor is, of necessity, an unsteady process. Adding to this the practical interest in oscillatory burning, it is clear that unsteady analysis has been long overdue. Since the complete unsteady problem presents presently insurmountable mathematical difficulties the various types of unsteadiness have been separately analysed. This separation, of necessity, also introduces errors, but errors which can be estimated. At least conditions can be stated where this separation is possible. It has therefore been possible to state certain criteria whereby droplets burn in a near steady state and when it is possible to consider periodic solutions of small amplitude taking place about this near steady state.

To obtain this steady state one criterion is that the liquid is not changing temperature in time; the statement must be that the heat-up time is a small fraction of the lifetime. For sufficiently low wet bulb temperatures compared to the ambient gas temperature, sufficiently small ratio of liquid specific heat to gas specific heat and sufficiently small vaporization rates this is true. Another criterion has been found to be the fact that initial conditions of the gas phase should relax to the near steady state conditions in a time short compared with the droplet lifetime. For a strong convective stream, small vaporization rate, large liquid to gas density ratio and sufficiently well behaved initial conditions this criteria is met. So that unsteadiness is not felt due to a contracting droplet radius and so that in the investigation of a periodic solution the droplet surface moves negligibly far during a cycle it is necessary that both the diffusion time and cycle time be much shorter than the lifetime. For a moderate enough convective stream, large liquid to gas density ratio and small transfer number this criterion is usually satisfied with the cycle times of the order of the diffusion time. The errors made are of the order of these time ratios when the steady state model is assumed valid. The periodic solution is asymptotically valid up to the order of these ratios.

Periodic solutions for what was called a "pressure effect" were obtained for some simple models of the vaporization process under somewhat artifical boundary conditions. These artifical conditions had to be introduced precisely because of the model simplicity; convective effects were not properly taken into account. It was seen that for frequencies commensurate with the diffusion time extremely strong response in the vaporization rate could be expected. In the limit of very high frequency it was found that the vaporization rate perturbation could become infinite as the square root of frequency but that the burning rate perturbations would remain bounded. With the aid of the simple plane model it was possible to investigate the effects of periodic liquid heat=up and finite evaporation kinetics. It was found that for many practical configurations these effects can be ignored.

In order to treat the convective effect properly the asymptotic limit of a high free stream Reynolds number flow over a liquid body was considered. What resulted was the investigation of the binary fluid boundary layer for the stagnation point and flat plate at low free stream Mach numbers. The condition for the existence of a flame at the stagnation point were investigated. It was found that only for sufficiently low Reynolds numbers, large droplet sizes, high temperatures, and fast reaction kinetics could the pressure be low enough from a practical point of view and still retain the collapsed flame zone assumption. It was found from the steady state theory that even with the collaped flame, a significant fraction of the vaporized mass is convected into the droplet wake rather than burned in the boundary layer.

It is perhaps best here to show that the theoretical results are plausible when compared with experimental correlations. Take the case Y_{Fw} = .9, τ = 0.2, and pure vaporization so that $Y_{O\infty}/j$ = 0. From Eq's. (1.25b and c) and the convective correlation, Eq. (1.27), the average vaporization rate per unit area is for high Re [1]

$$m = 0.3 \ln(1+B)\, Re^{1/2} = 0.69\, Re^{1/2}$$

Putting the stagnation point results in the same non-dimensional form ($m = \frac{\rho^* v^*}{\mu_\infty}$) there results from Eq. 3.62 and Table 3

$$m = 0.97\, Re^{1/2}$$

which is, as expected, higher than the average over the body. Apparently, therefore, the flat plate and stagnation point results can be empirically matched to obtain the correct total vaporization rate at high Reynolds numbers.

A periodic solution was then obtained to the linearly perturbed problem by introducing a longitudinally travelling isentropic sound wave of arbitrarily small amplitude in the free stream. This type of wave was chosen for analytic simplicity; however, because of the

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1. Re on this page is based on droplet diameter.

linearity any wave type may be built by superposition of these. In fact, in any practical device such as longitudinal wave must invariably be a standing wave in order to satisfy the combustor boundary conditions. It was found possible to solve the problem for arbitrary free stream Mach number by splitting the problem into two parts; one was primarily to compression of the film and was called a pressure effect analogous to what was studied for the simpler models and the second was called a velocity effect, primarily due to the changing convective field. However, it should be cautioned that this is in reality a mathematical artifice and the distinction is somewhat artifical; it happens solely because this is a compressible fluid with the waves propagating at a finite speed through the ambient gas (the characteristics on an x - t diagram have a finite slope).

Striking similarities were found between the flat plate and the stagnation point results for both high and low frequency. It can therefore be conjectured, and there is no theoretical justification for this, that the qualitative behavior of the flat plate can be constructed from a knowledge of the low and high frequency results for the flate plate. and the entire spectrum for the stagnation point. Figure 30 represents this conjecture. The low frequency end of the velocity effect is constructed from the quasi-steady and first order in frequency results using the stagnation point observation that real quantities retain the quasi-steady results and imaginary quantities are nearly linear out to ω of $O[1]$. The high frequency results use the asymptotic high frequency results for the flat plate. The mid-frequency range should, of course, only be interpreted as a rough guess based on the two endpoints and the stagnation point behavior. The pressure response is constructed in a similar manner; only real quantities are presented.

Then consider a droplet as acting partly as a flat plate and partly as a stagnation point. Consider first the pressure effect for the fundamental mode so that the droplets are considered very close to a velocity node. With increasing frequency (shortening the chamber length) the vaporization rate component in phase with the pressure will slowly decrease and finally move 180° out of phase with the pressure at $\omega \sim 2$.[1] A strong negative peak will then develop as the frequency is further increased. Only at very high frequencies will the response become positive again. The point of the first sign change can be strongly and practically controlled by a change in $Y_{O\infty}/j$. The magnitude of the quasi-steady results should be between .25 and 1.25 when normalized by the steady state vaporization rate; these are generally the magnitudes of the stagnation point and

1. Possibly the flat plate shift does not occur until $\omega \sim 4$.

flat plate, respectively.[1] The same behavior is noted for the real part of the mass burning rate perturbation with one important exception: for normal values of T_f and γ the burning rate will never return to being in phase with the pressure once a sufficiently high frequency level is reached. The point of the sign change is most strongly influenced by $Y_{0\infty}/j$ and q. Also, for sufficiently high $Y_{0\infty}/j$ and presumably for q a small peak will develop in the response at low frequency in phase with the pressure. The magnitude of this response is usually approximately 1.

Continuing, the velocity effect is added. It should be clear that shortening the chamber length must bring in this effect if the droplet location is fixed. Therefore, at high enough frequencies in practical configurations this effect must enter. For the fundamental mode for sufficiently high $Y_{0\infty}/j$ or q it is seen that there can develop a rather strong peak for the burning rate in phase with the pressure at a rather low frequency.[2] The magnitude and frequency point of the maximum must depend strongly upon how much a droplet acts as a stagnation point and how much as a flat plate. It must also be primarily strongly dependent upon $Y_{0\infty}/j$. It is therefore clear that combining both the pressure and velocity effects can produce a reasonably strong peak at low frequencies in phase with the pressure for the burning rate. It is easily conceivable that the

1. This may easily be computed from Figures 7, 9, 12, and 20.
2. Recall that a velocity effect is much stronger than a pressure effect.

magnitude of this combined response can become much greater than one relative to the pressure effect alone. This type of response requires that droplets are sufficiently far from a velocity node. The velocity effect of the vaporization rate perturbation is opposite for the flat plate and stagnation point and little change from the pressure effect alone should be expected, at least for the fundamental. Recall, however, that construction of $P_{i_{(e)}}(o)$ may be strongly in error for the flat plate past ω of 0 [1] and the strong positive peaking for the stagnation point may be dominant, causing a strong augmentation of the negative pressure effect peak.

Considering the higher harmonics, recall that real parts of the velocity effect can be in phase with the pressure. These are extremely strong at the low frequency end (the quasi-steady results). Both the vaporization rat and burning rate are augmented in phase with the pressure except at very high frequencies for the vaporization rate. Opposite effects come into play with respect to frequency;[1] 1.) raising the harmonic increases the component in phase with pressure but decreases the response magnitude and 2.) increasing the frequency for a given harmonic decreases the response but would also move the droplet position with respect to the wave position so as to increase the response.

If then, consideration is given to the problem of increaseing a chamber length where initially, say, $\omega \sim 5$ the following sequence can be expected for

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1. Here the droplet position is held constant with respect to the acoustic chamber.

the burning rate components in phase with the pressure for sufficiently high $Y_{o\infty}/j$ and q.

1. There is a positive response for the fundamental, out of phase with the pressure. Consider only those harmonics which still allow the droplets to be on the left hand side of the first pressure node so that $\sin n\pi X$ has the same sign as $\cos n\pi X$. Then the combined pressure and velocity effects favor the lower harmonics; the higher the harmonics the less chance it has of being in phase with the pressure. It is possible, however, that the first few have a positive response.

2. As the frequency decreases the fundamental response will go in phase with the pressure. The normalized response can easily reach O [1].[1] The fundamental will undergo a maximum response and then settle to the quasi-steady pressure response. More harmonics will be brought into play.

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1. This is the known magnitude of response necessary to overcome chamber damping in a liquid propellant rocket engine. See References 29 and 30.

3. The components for each harmonic brought into play will become stronger in phase with pressure. This response can easily be much greater than 0 [1], approaching 0 [1/M]. The second harmonic will undergo a maximum response due to the effective movement of droplets toward a velocity node.
4. The successively higher harmonics will go through approximately the same history as the second but with increased reliance on the real part of $M_{F_f} / \bar{m}_{F_f}$ and the movement towards a velocity node as determining factors since the velocity effect is most important for the higher harmonics.

Concerning the vaporization rate, substantially the same behavior will occur for the higher harmonics as for the burning rate. If, however, the stagnation point behavior is sufficiently strong the fundamental will depend primarily on the pressure effect alone and no strong maximum can be expected. Here comment is restricted to $\omega \gtrsim 5$. The reason that higher frequencies are now excluded is that they do not exist in practice for drop sizes of interest (75-500 μ) in high performance combustors.

In summary for the above, it has been seen that the burning rate perturbation can undergo striking changes with frequency for the assumed wave type if one assumes a droplet acts partly as a flat plate and partly as a stagnation point. The most important parameters in determining this behavior are q and Y_{O_∞}/j. The velocity effect is of paramount importance to this behavior. Conclusions concerning other wave types, quantities of interest, liquid body configurations, or frequency levels can easily be drawn from the conclusions of Chapter IV. The primary type of response which stimulated this entire work was, however, that immediately investigated above.

Definite conclusions concerning actual combustor performance have been avoided as much as possible. There are several reasons for this and they are linked to areas in which further research should take place. First, in an actual combustor the droplets are not at a fixed location but are moving at a speed of the same order as the gas speed, usually. Secondly, the free stream relative to the droplets is not a constant one nor is the mean flow constant with respect to the chamber since gas is continually being evolved; the relative speed effect has been discussed already and generally been found to be unimportant as long as a high Reynolds number flow exists throughout most of the droplet lifetime , but the changing mean flow effect has not been treated. Also, the effect of other boundary condition gradients has not been treated.

Thirdly, actual average steady state ambient boundary conditions are not known with any precision and, in many cases, not even roughly. Under this effect can also be included the fact that the relative direction of the mean flow over the droplet can easily change once during its lifetime if at the injection point the injection speed is higher than the gas speed. Fourth, in many cases it is not possible to justify the existence of a collapsed flame or even burning in the forward stagnation region. The fifth point, not even touched upon, is the effect of turbulence. Sixth, the slip effect at the droplet surface may be important. Seventh, the droplet can easily undergo distortion oscillations of the same frequency as considered. Eighth, research should be done on transverse perturbations about a longitudinal mean flow since transverse oscillations are quite important in practice. This represents a much more difficult problem since a transverse perturbation would essentially be a low Reynolds number cross-flow, time dependent. Finally, and probably most important, a great deal of mass is carried into the wake and not burned in the leading edge boundary layer; therefore, wake flames deserve to be treated in the unsteady state. It is worthy of note that the typical diffusion time for such a problem would be much longer, a^* being the characteristic length rather than $\delta^* \sim a^*/\sqrt{Re}$. Therefore higher dimensionless frequencies are practically possible.

The remarkable fact is, however, that a blind application of the above theory is observed to explain many of the qualitative features of longitudinal rocket engine instability (33) if the burning rate perturbation is accepted as the appropriate feedback function. Work concerning this application should continue.

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TABLE I

INITIAL CONDITION RELAXATION TIME, t^*_{IC}/t^*_{dif}

(Initial Condition: $y_o(r) = 1$)

r_f/r_L \ B	5	10	15	20
1.5	.0388	.0380	.0372	.0364
2.0	.1446	.1530	.1489	.1425
3.0	.6170	.6054	.5964	.5860
5.0	2.3880	2.4600	2.4615	2.4260
10.0	13.1050	12.6600	12.0615	11.9340

TABLE 2

FLAT PLATE - STEADY STATE

	Y_{F_w}	0	0.25	0.5	0.7	0.8	0.9	0.95
Y_{o_∞}/j								
0	$F(0)$	0	-.12674	-.27907	-.43317	-.53010	-.65385	-.73686
Pure Vaporization Limit	$F''(0)$	.46960	.38008	.27901	.18562	.13251	.07264	.03878
	ζ_f	Blasius	∞					
	F_f							
	F'_f		1					
	F''_f		0					
	$\sqrt{2X}\,\bar{m}F_f$		0					
0.05	$F(0)$	-.02266	-.14660	-.29547	-.44592	-.54041	-.66086	-.74148
	$F''(0)$	.45327	.36648	.26861	.17837	.12716	.06956	.03707
	ζ_f	0	2.32012	3.04840	3.67120	4.10737	4.81791	5.52374
	F_f	-.02266	.87591	1.17224	1.30117	1.34800	1.38671	1.40329
	F'_f	0	.83333	.90909	.93333	.94118	.94737	.95
	F''_f	.45327	.23164	.14966	.11721	.10582	.09646	.09242
	$\sqrt{2X}\,\bar{m}F_f$	.02266	.06949	.08231	.08791	.08995	.09164	.09242
0.15	$F(0)$	-.06326	-.18250	-.32511	-.46889	-.55897	-.67344	-.74975
	$F''(0)$	.42410	.34219	.25008	.16549	.11768	.06414	1.03408
	ζ_f	0	1.68182	2.45562	3.10614	3.55281	4.27458	4.98725
	F_f	-.06362	.33894	.59723	.72429	.77232	.81273	.83036
	F'_f	0	.625	.76923	.82353	.84211	.85714	.86364
	F''_f	.42410	.34837	.26726	.22388	.20689	.19212	.18561
	$\sqrt{2X}\,\bar{m}F_f$	.06362	.13935	.17372	.19029	.19654	.20181	.20417
0.3	$F(0)$	-.11621	-.22855	-.36302	-.49819	-.58258	-.68935	-.76017
	$F''(0)$	.38735	.31166	.22689	.14946	.10592	.05745	.03041
	ζ_f	0	1.29403	2.06915	2.73325	3.18762	3.91864	4.63782
	F_f	-.11621	.05645	.24247	.34889	.39139	.42808	.44432
	F'_f	0	.45455	.625	.7	.72727	.75000	.76
	F''_f	.38735	.37131	.32252	.28694	.27121	.25678	.25006
	$\sqrt{2X}\,\bar{m}F_f$	.11621	.20422	.25802	.28694	.29833	.30814	.31257
0.5	$F(0)$	-.17396	-.27903	-.40447	-.53008	-.60819	-.70653	-.77136
	$F''(0)$	.34793	.27903	.20224	.13252	.09357	.05047	.02660
	ζ_f	0	1.04467	1.79821	2.46658	2.92590	3.66446	4.38978
	F_f	-.17396	-.11180	.00717	.08705	.12098	.15115	.16471
	F'_f	0	1/3	1/2	.58333	.61539	.64286	.65517
	F''_f	.34793	.35279	.33110	.30869	.29739	.28632	.28096
	$\sqrt{2X}\,\bar{m}F_f$	.17396	.26459	.33110	.37043	.38661	.40085	.40739

TABLE 3

STAGNATION POINT - STEADY STATE

$q = 12 \qquad \tau = 0.2$

Y_{o_∞}/j	Y_{F_W}	0	0.25	0.5	0.7	0.8	0.9	0.95
0	$F(0)$						-.68460	-.80730
	$F''(0)$						.24023	.18176
	η_f						∞	∞
	F_f							
	F'_f						1	1
	F''_f							
	T_f						1	1
	B_∞						11.25	23.75
	^mF_f						0	0
0.05	$F(0)$	-.02114	-.14698	-.30434	-.47288	-.58641	.74657	.87336
	$F''(0)$	.73703	.76845	.63091	.48451	.39278	.27781	.20267
	η_f	0	1.76409	2.28211	2.70089	2.97878	3.40802	3.80750
	F_f	-.02114	.87371	1.15382	1.27702	1.32315	1.36320	1.38253
	F'_f	0	.96488	1.02357	1.04025	1.04546	1.04962	1.05148
	F''_f	.73703	.15753	.03713	-.00608	-.02102	-.03363	-.03951
	T_f	.2	1.36667	1.47273	1.50667	1.51765	1.52632	1.53
	B_∞	.03573	.28571	.78571	1.78571	3.03571	6.78571	14.2857
	^mF_f	.04227	.13237	.15628	.16689	.17088	.17435	.17603
0.15	$F(0)$	-.05952	-.19481	-.36362	-.54485	-.66672	-.83735	-.96100
	$F''(0)$	.70232	.90371	.77371	.60050	.48495	.33548	.23593
	η_f	0	1.19419	1.68443	2.07435	2.33014	2.72455	3.09368
	F_f	-.05952	.40223	.68791	.82889	.88361	.93189	.95515
	F'_f	0	.89514	1.06330	1.11993	1.13871	1.15409	1.16102
	F''_f	.70232	.38586	.13782	.02307	-.02048	-.05880	-.07715
	T_f	.2	1.82500	2.2	2.34118	2.38474	2.42857	2.44545
	B_∞	.05770	.20516	.5	1.08974	1.82692	4.03846	8.46154
	^mF_f	.11904	.28666	.36023	.39617	.41010	.42240	.42832

TABLE 3

STAGNATION POINT - STEADY STATE

$q = 12 \qquad \tau = 0.2$

	Y_{F_w}	0	0.25	0.5	0.7	0.8	0.9	0.95
Y_{o_∞}/j								
0.3	F(0)	-.10925	-.25607	-.43598	-.62852	-.75770	-.93733	-1.07528
	F"(0)	.65819	1.00833	.90929	.71817	.58082	.39708	.27231
	ζ_f	0	.87205	1.31606	1.67377	1.90818	2.27029	2.61063
	F_f	-.10925	.11580	.35808	.49706	.55403	.60559	.63065
	F'_f	0	.79656	1.07960	1.19434	1.23523	1.27001	1.28589
	F''_f	.65819	.61727	.31916	.13795	.06165	-.00944	-.04425
	T_f	.2	2.2	2.95	3.28	3.4	3.5	3.544
	B_∞	.06819	.16667	.36364	.75758	1.25	2.72727	5.68182
	$\tilde{m}F_f$	.21851	.44643	.58085	.65353	.68296	.70952	.72236
0.5	F(0)	-.16453	-.32467	-.51507	-.71695	-.85177	-1.03827	-1.17979
	F"(0)	.61031	1.07565	1.01980	.82079	.66629	.45313	.30583
	ζ_f	0	.67629	1.07470	1.40332	1.61965	1.95470	2.27134
	F_f	-.16453	-.07940	.10082	.22290	.27622	.32597	.35066
	F'_f	0	.69523	1.06478	1.24092	1.30817	1.36714	1.39508
	F''_f	.61031	.80153	.53545	.32003	.21895	.11985	.06859
	T_f	.2	2.46667	3.6	4.16667	4.38462	4.57143	4.65517
	B_∞	.07353	.14706	.29412	.58824	.95588	2.05882	4.26471
	$\tilde{m}F_f$	.32907	.60572	.79931	.91394	.96235	1.00697	1.02890

TABLE 3

STAGNATION POINT - STEADY STATE

$Y_{O_\infty}/j = 0.15 \qquad Y_{F_w} = 0.9$

τ	q	0	2	4	6	8	10	12
0.2	$F(0)$	-.71140	-.73762	-.76109	-.78246	-.80212	-.82036	-.83735
	$F''(0)$	.22629	.24562	.26443	.28277	.30070	.31826	.33548
	ζ_f	3.19164	3.08126	2.98878	2.90956	2.84047	2.77931	2.72455
	F_f	.78401	.81509	.84265	.86773	.89079	.91212	.93189
	F'_f	.91556	.96196	1.00501	1.04532	1.08336	1.11951	1.15409
	F''_f	.15100	.11321	.07676	.04151	.00731	-.02605	-.05881
	T_f	.88571	1.14286	1.4	1.65714	1.91429	2.17143	2.42857
	B_∞	13.12500	9.54546	7.5	6.17647	5.25	4.56522	4.03846
	m_{F_f}	.37206	.38243	.39178	.40032	.40821	.41555	.42240

q	τ	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
12	$F(0)$	-.80841	-.82333	-.83735	-.85053	-.86292	-.87480	-.89684	-.91712
	$F''(0)$	.25499	.29584	.33548	.37404	.41166	.44844	.51979	.58861
	ζ_f	2.79029	2.75599	2.72455	2.69549	2.66838	2.64317	2.59706	2.55583
	F_f	.91854	.92528	.93189	.93822	.94411	.95011	.96128	.97186
	F'_f	1.14586	1.15005	1.15409	1.15809	1.16219	1.16604	1.17382	1.18142
	F''_f	-.04222	-.05067	-.05881	-.06686	-.07511	-.08287	-.09855	-.11390
	T_f	2.4	2.41429	2.42857	2.44286	2.45714	2.47143	2.5	2.52857
	B_∞	3.75	3.88889	4.03846	4.2	4.375	4.56522	5	5.52632
	m_{F_f}	.41877	.42060	.42240	.42413	.42576	.42740	.43049	.43344

Coordinate systems and geometrical configurations

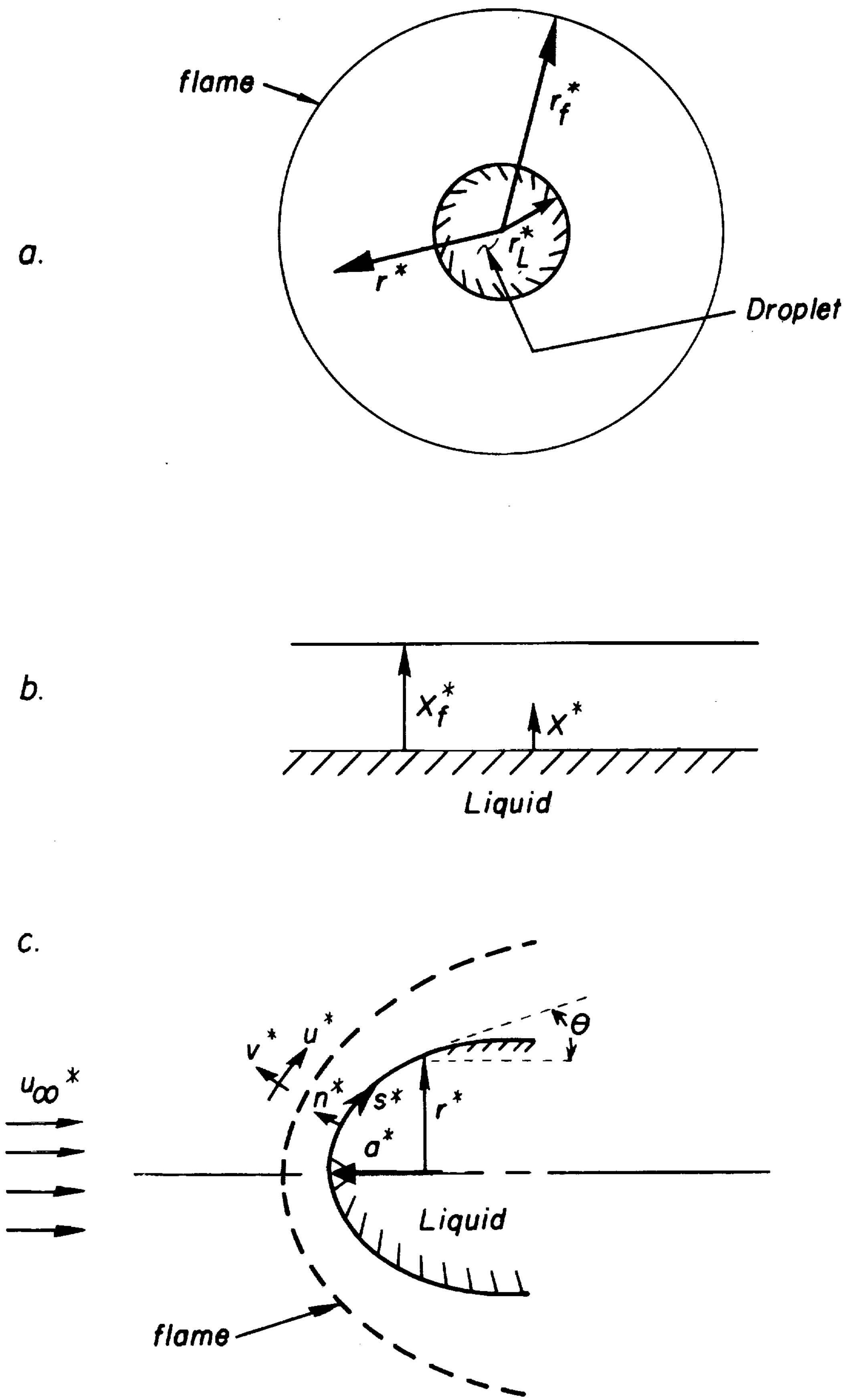
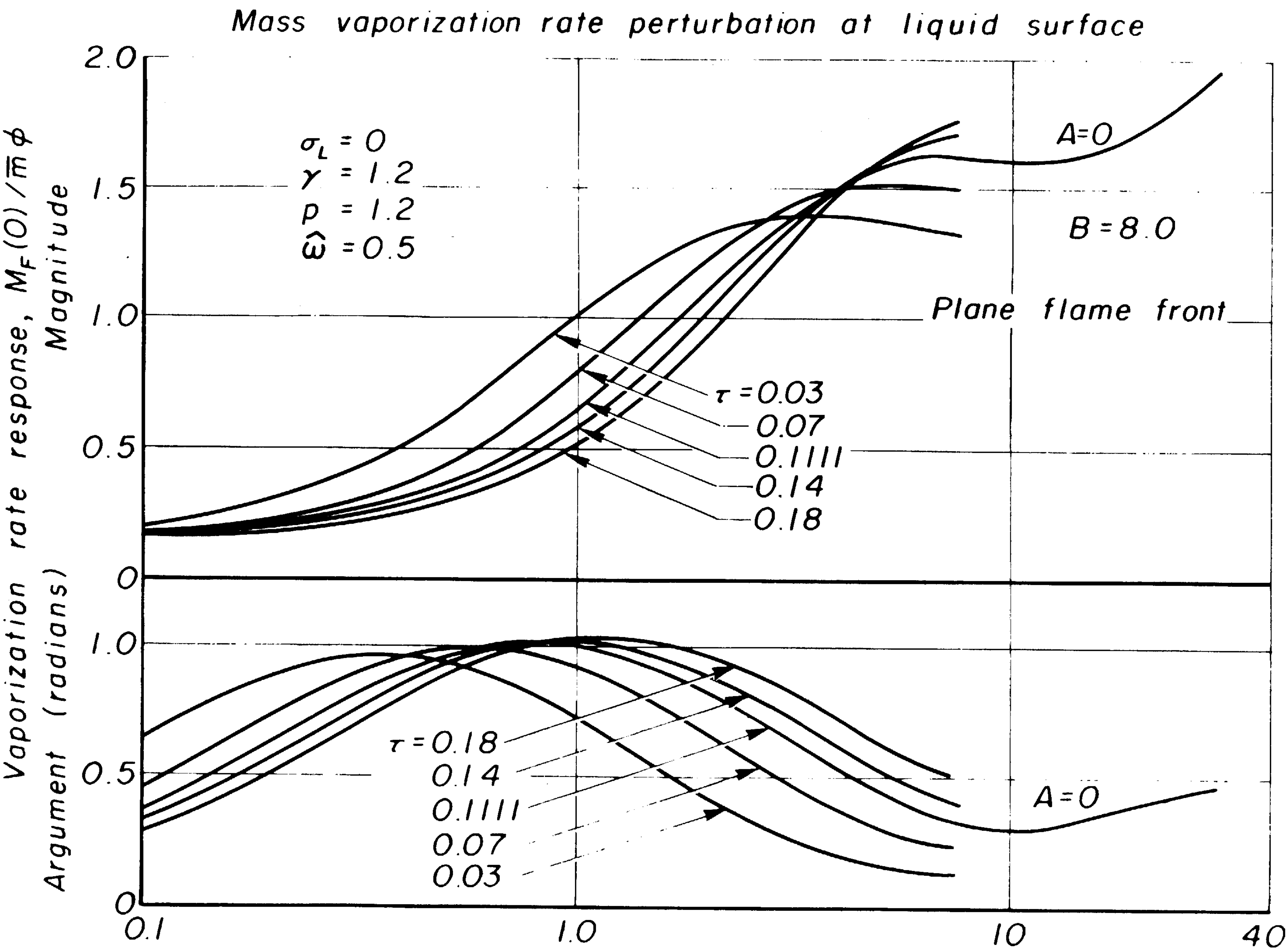
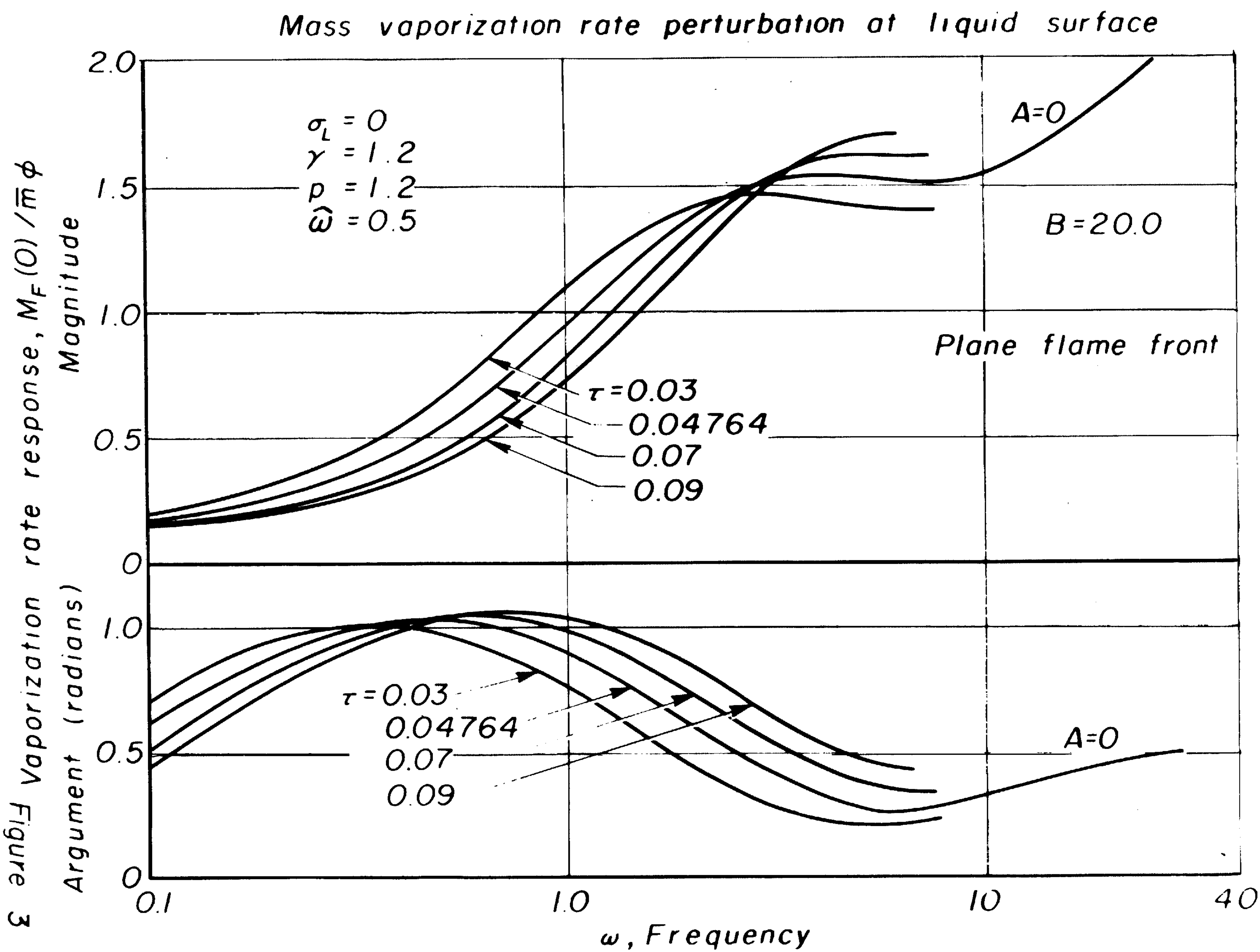


Figure 1



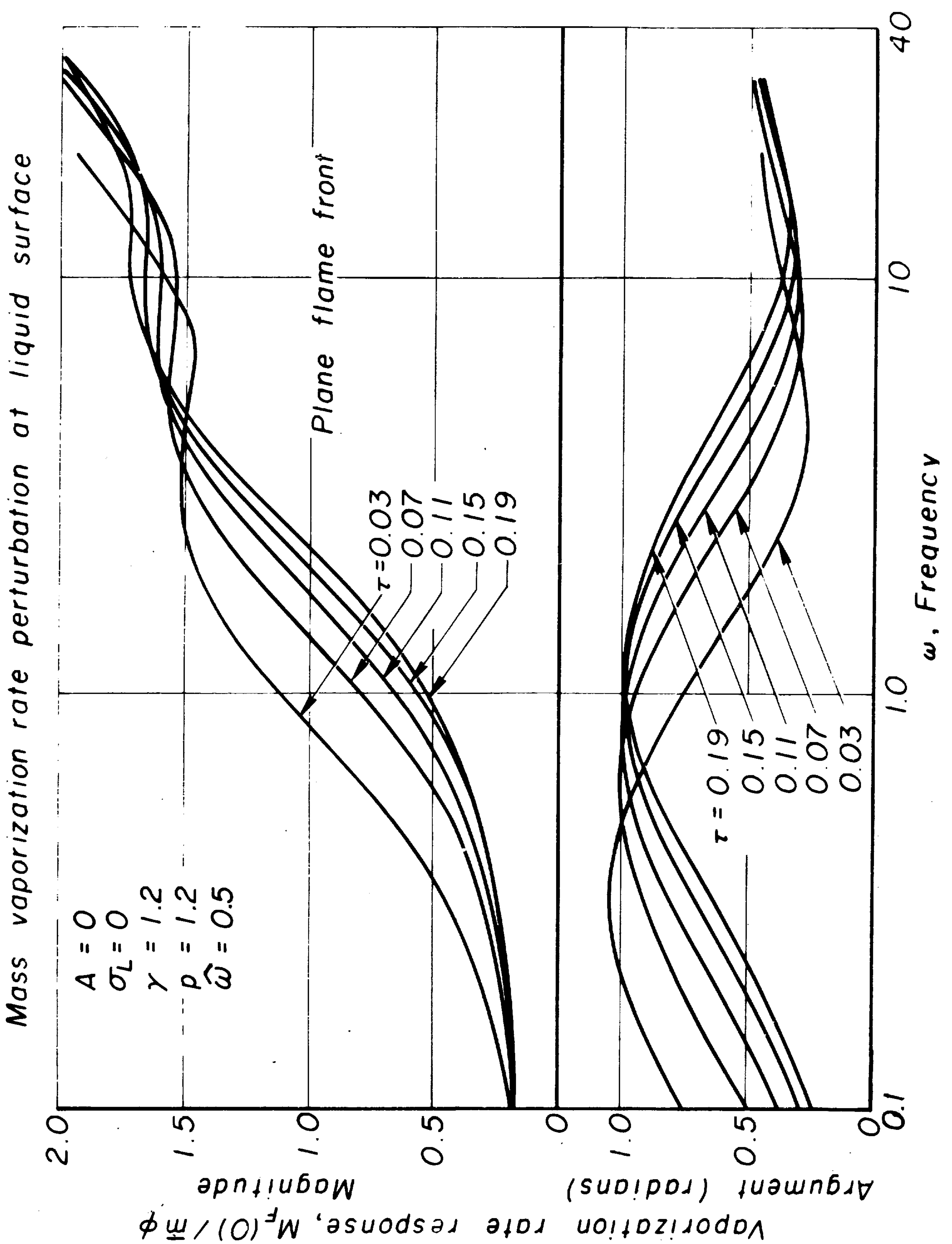
Mass vaporization rate perturbation at liquid surface
Vaporization rate response, $M_F(0)/\bar{m}\phi$
Magnitude
Argument (radians)
$\sigma_L = 0$
$\gamma = 1.2$
$p = 1.2$
$\hat{\omega} = 0.5$
A=0
B=8.0
Plane flame front
$\tau = 0.03$
0.07
0.1111
0.14
0.18
$\tau = 0.18$
0.14
0.1111
0.07
0.03
A=0
2.0
1.5
1.0
0.5
0
1.0
0.5
0
0.1
1.0
10
40
ω, Frequency

Figure 2



Mass vaporization rate perturbation at liquid surface
Vaporization rate response, $M_F(0)/\bar{m}\phi$
Magnitude
Argument (radians)
$\sigma_L = 0$
$\gamma = 1.2$
$p = 1.2$
$\hat{\omega} = 0.5$
A=0
B=20.0
Plane flame front
τ=0.03
0.04764
0.07
0.09
τ=0.03
0.04764
0.07
0.09
A=0
2.0
1.5
1.0
0.5
0
1.0
0.5
0
0.1
1.0
10
40
ω, Frequency

Figure 3



Mass vaporization rate perturbation at liquid surface
Plane flame front
A = 0
$\sigma_L = 0$
$\gamma = 1.2$
p = 1.2
$\hat{\omega} = 0.5$
τ = 0.03
0.07
0.11
0.15
0.19
τ = 0.19
0.15
0.11
0.07
0.03
ω, Frequency
0.1
1.0
10
40
Magnitude
Argument (radians)
Vaporization rate response, $M_F(0)/\bar{m}\phi$
2.0
1.5
1.0
0.5
0
1.0
0.5
0

Figure 4

Effect of liquid heat transfer on vaporization rate response

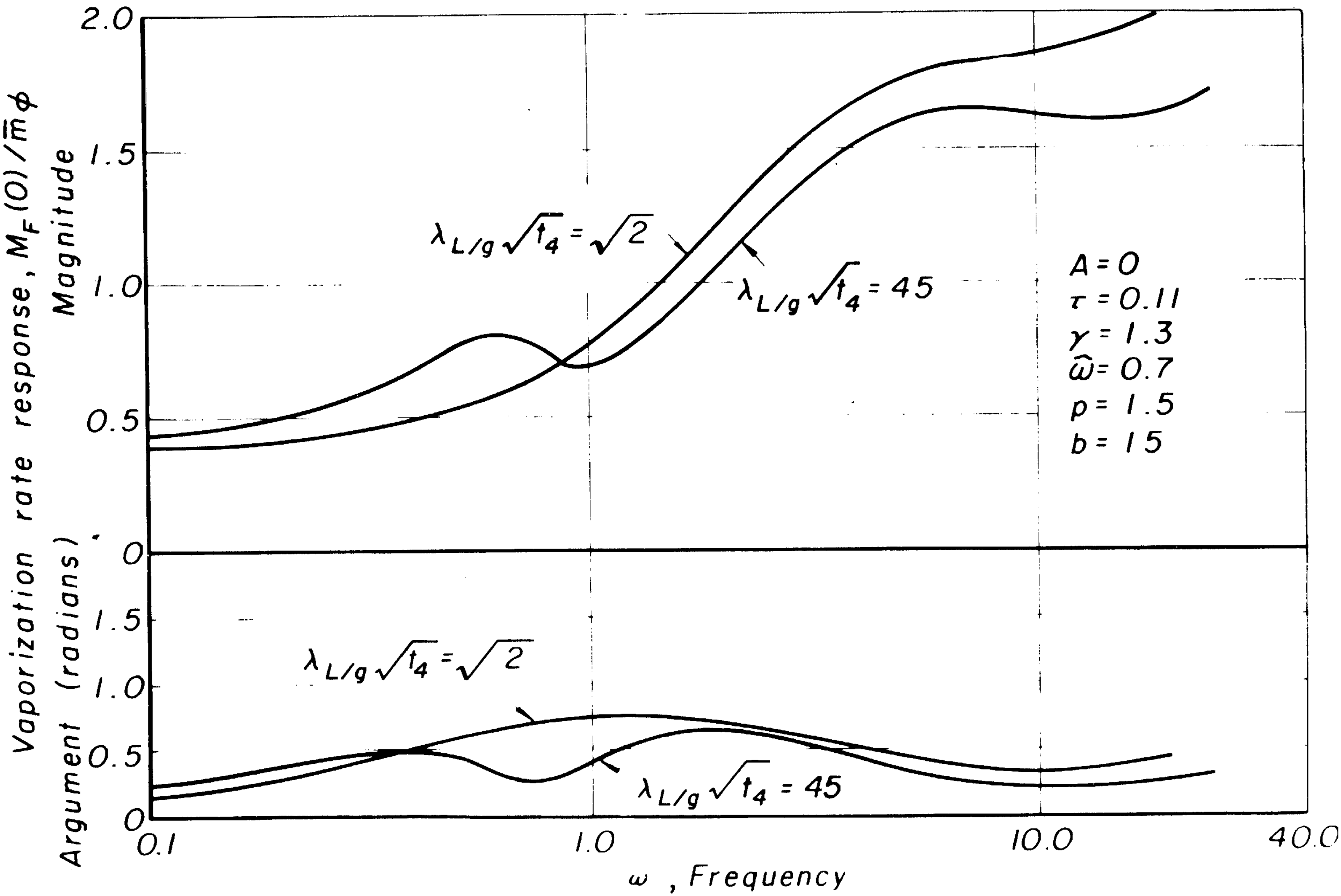


Figure 5

Effect of surface kinetics on vaporization rate response

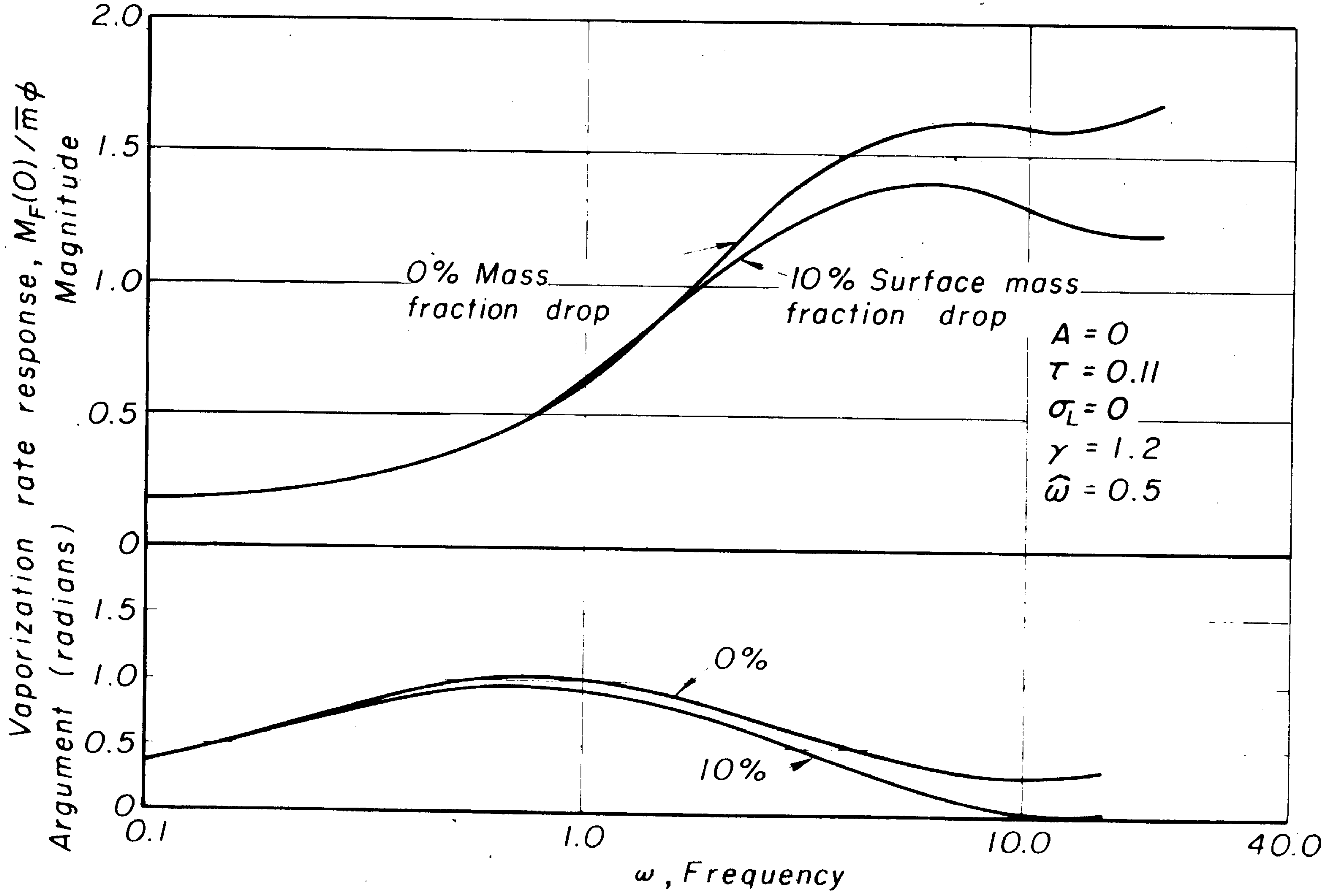
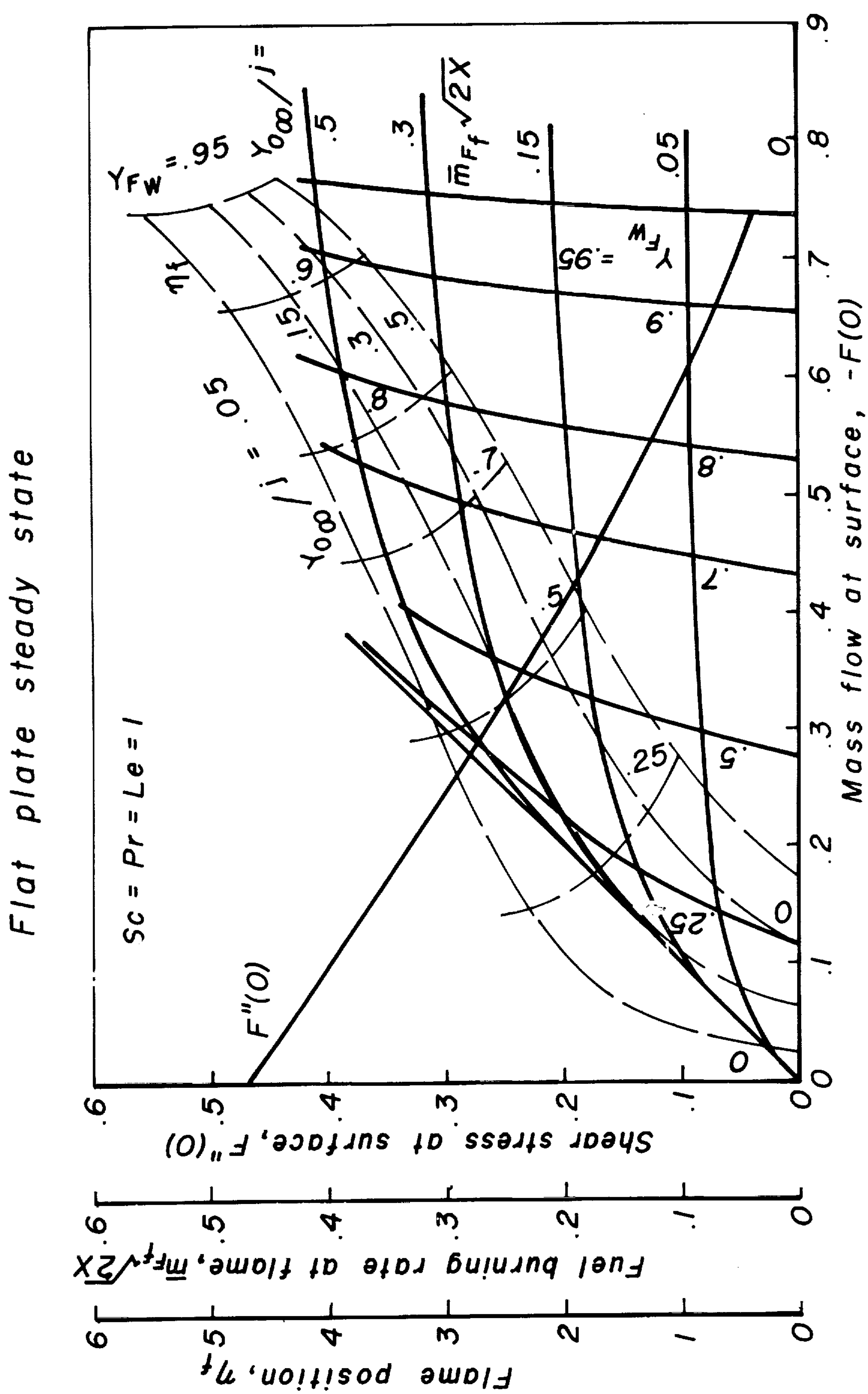


Figure 6



Flat plate steady state
Sc = Pr = Le = 1
Shear stress at surface, F''(0)
Fuel burning rate at flame, $\bar{m}_{F_f}\sqrt{2X}$
Flame position, η_f
Mass flow at surface, -F(0)
Y_{Fw} = .95
$Y_{O\infty}/j$ =
$Y_{O\infty}/j$ = .05
F''(0)
η_f

Figure 7

Flat plate steady state
Velocity and shear stress profiles

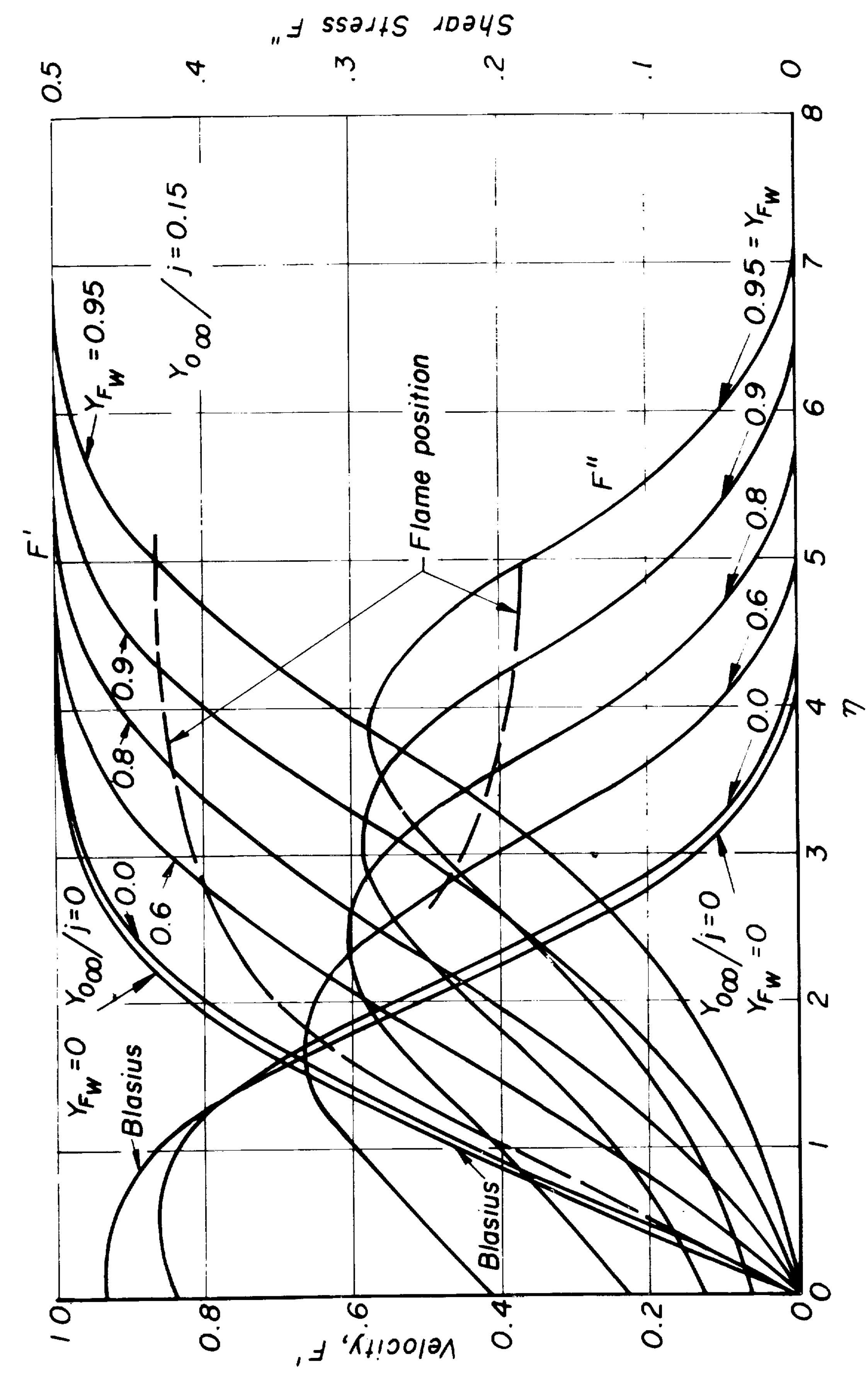
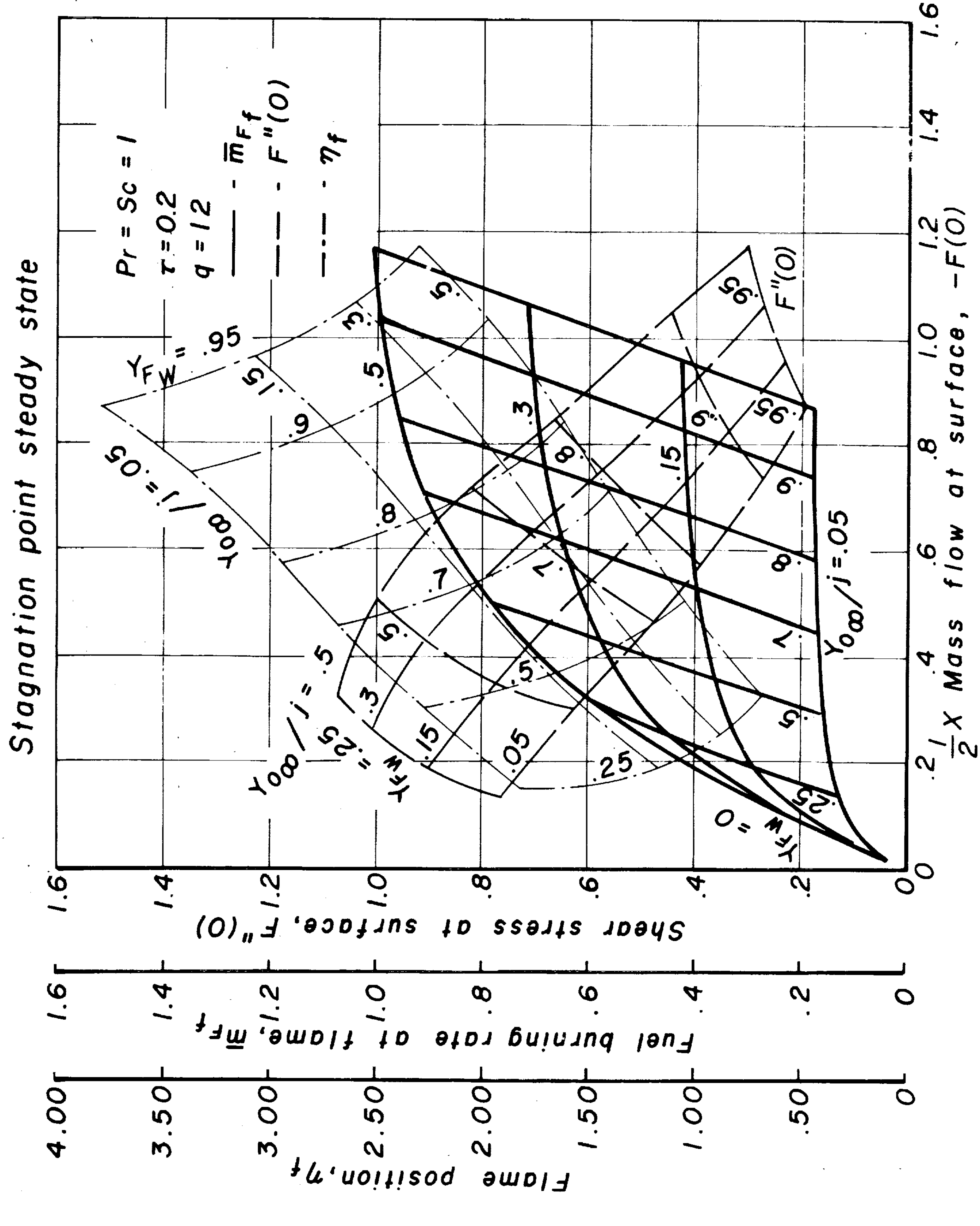


Figure 8



Stagnation point steady state
Pr = Sc = 1
τ = 0.2
q = 12
— $\bar{m}_{F_f}$
— F''(0)
— η_f
Y_{FW} = .95
Y_{FW} = .25
Y_{FW} = 0
$Y_{O\infty}/j$ = .05
$Y_{O\infty}/j$ = .5
$\frac{1}{2}$ X Mass flow at surface, −F(0)
Shear stress at surface, F''(0)
Fuel burning rate at flame, $\bar{m}_{F_f}$
Flame position, η_f

Figure 9

Stagnation point steady state

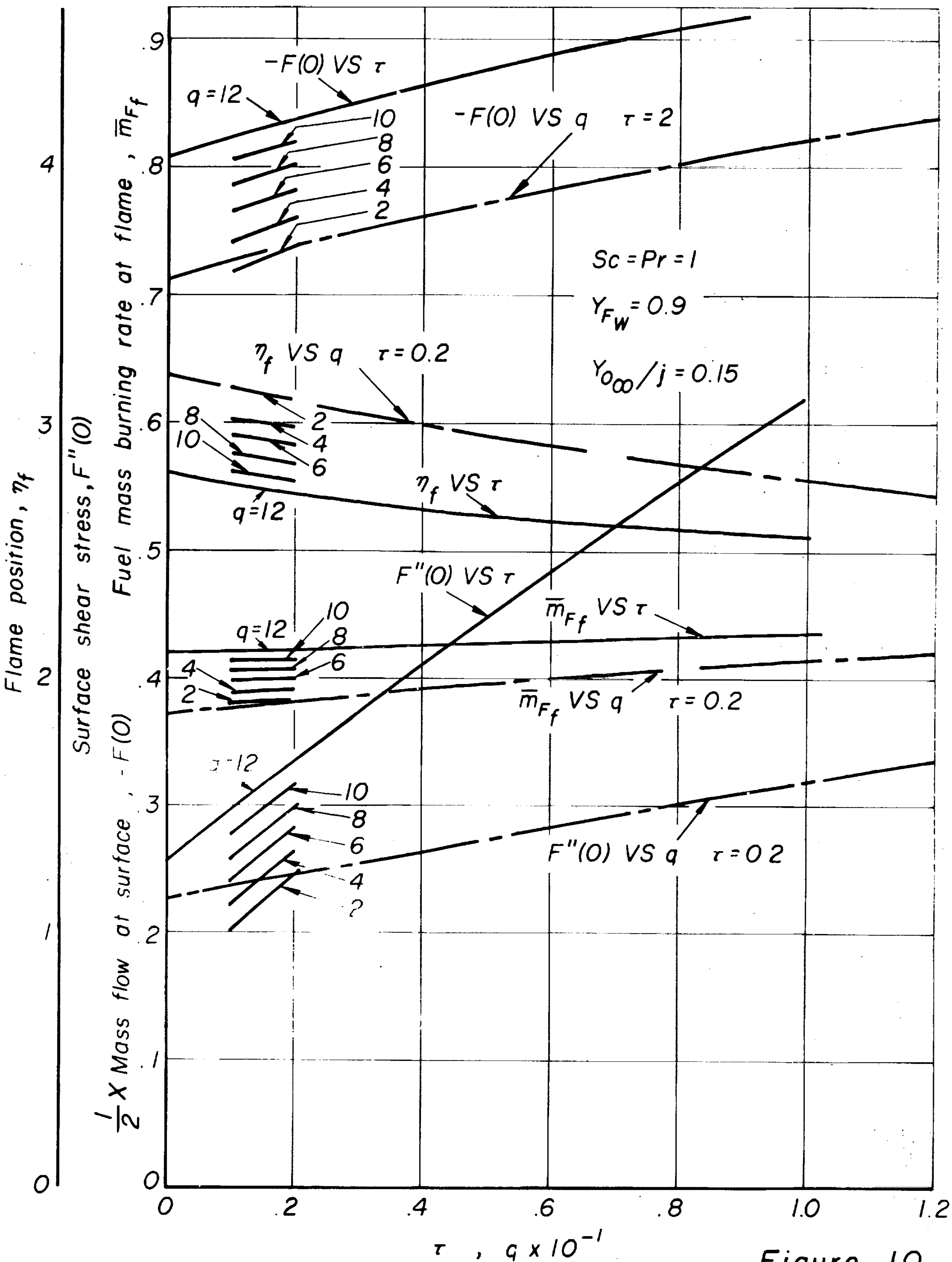


Figure 10

Stagnation point - steady state
Velocity and temperature profiles

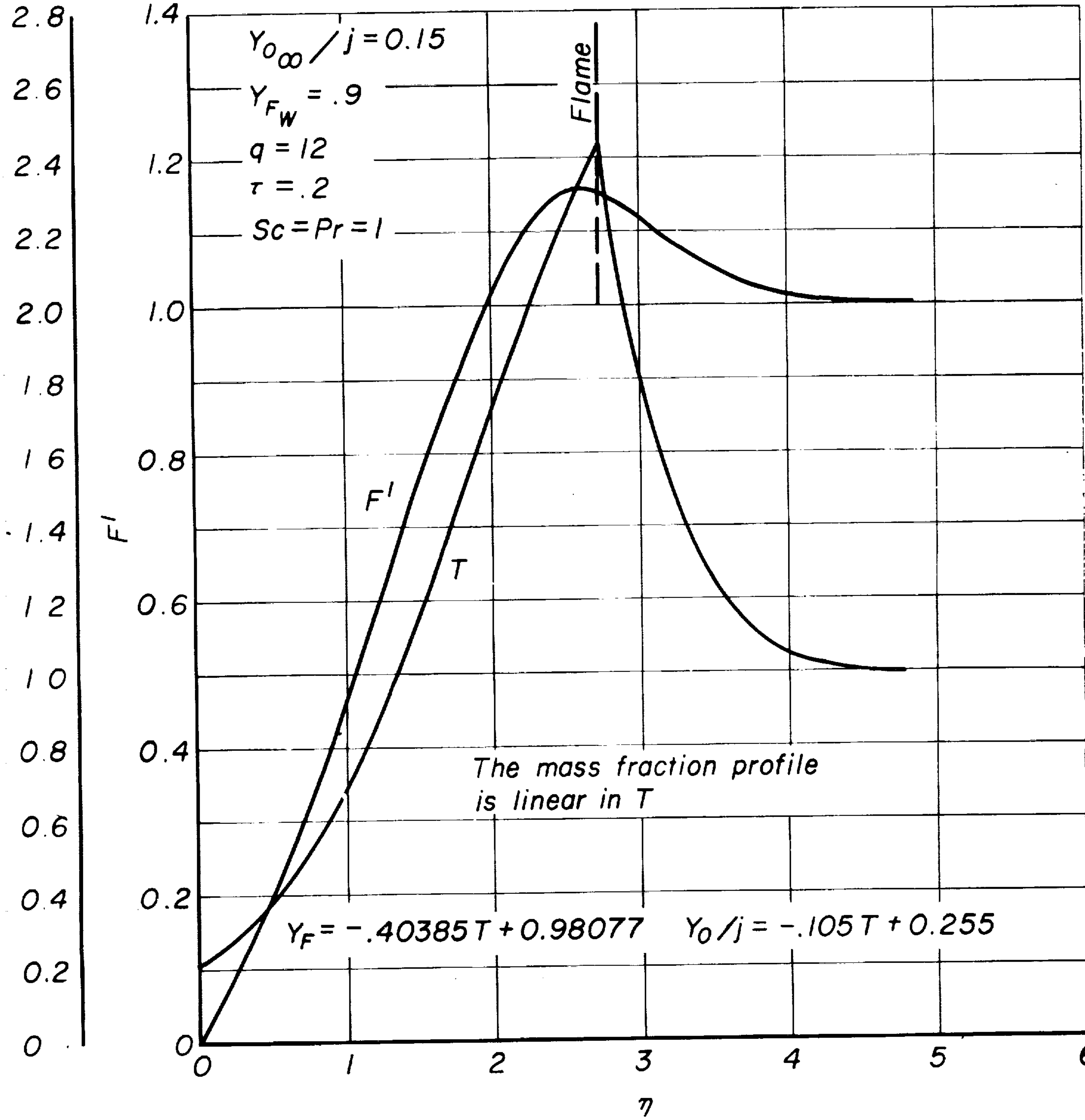


Figure 11

Quasi steady pressure response of flat plate boundary layer

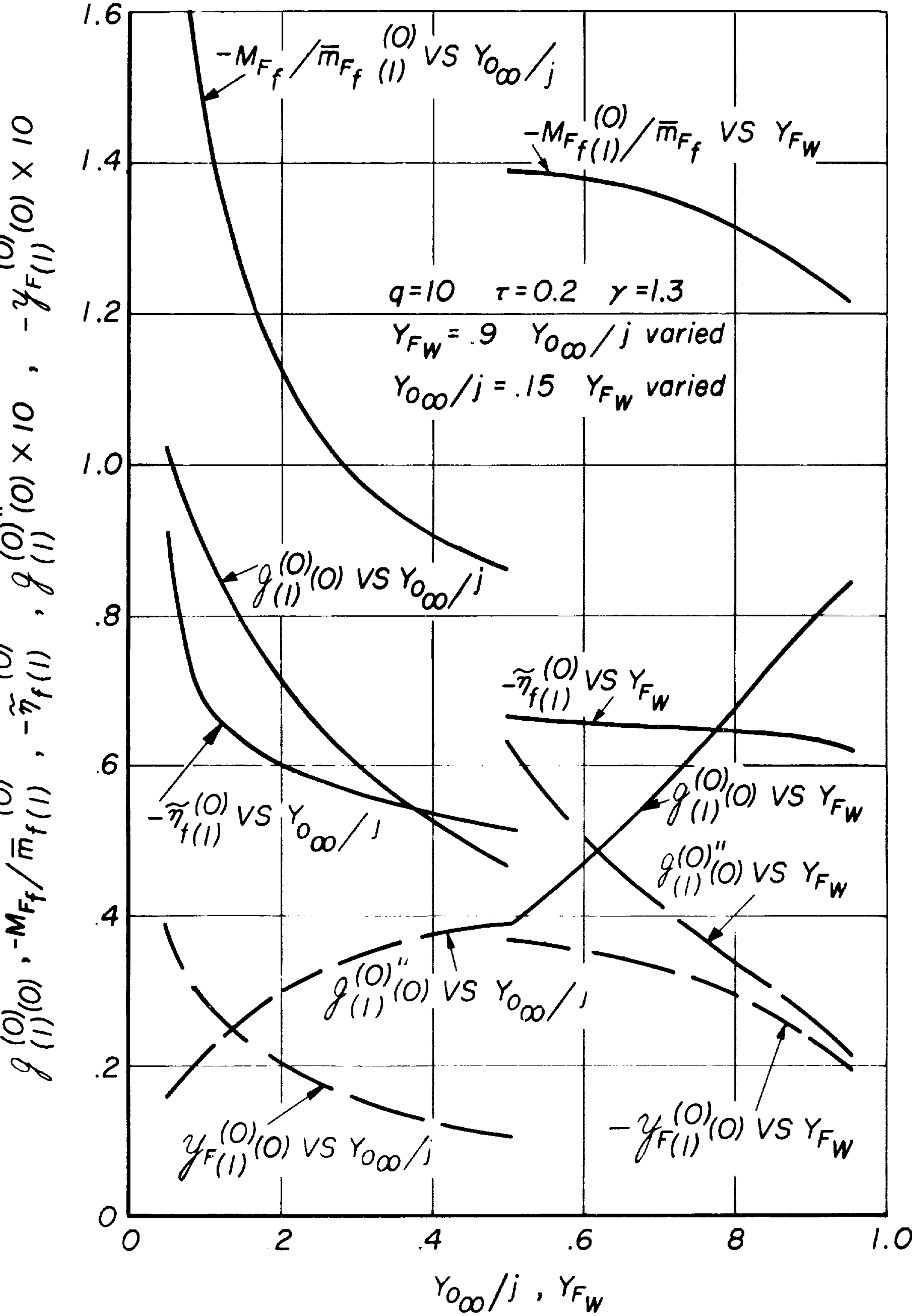


Figure 12

Quasi steady pressure response of flat plate boundary layer

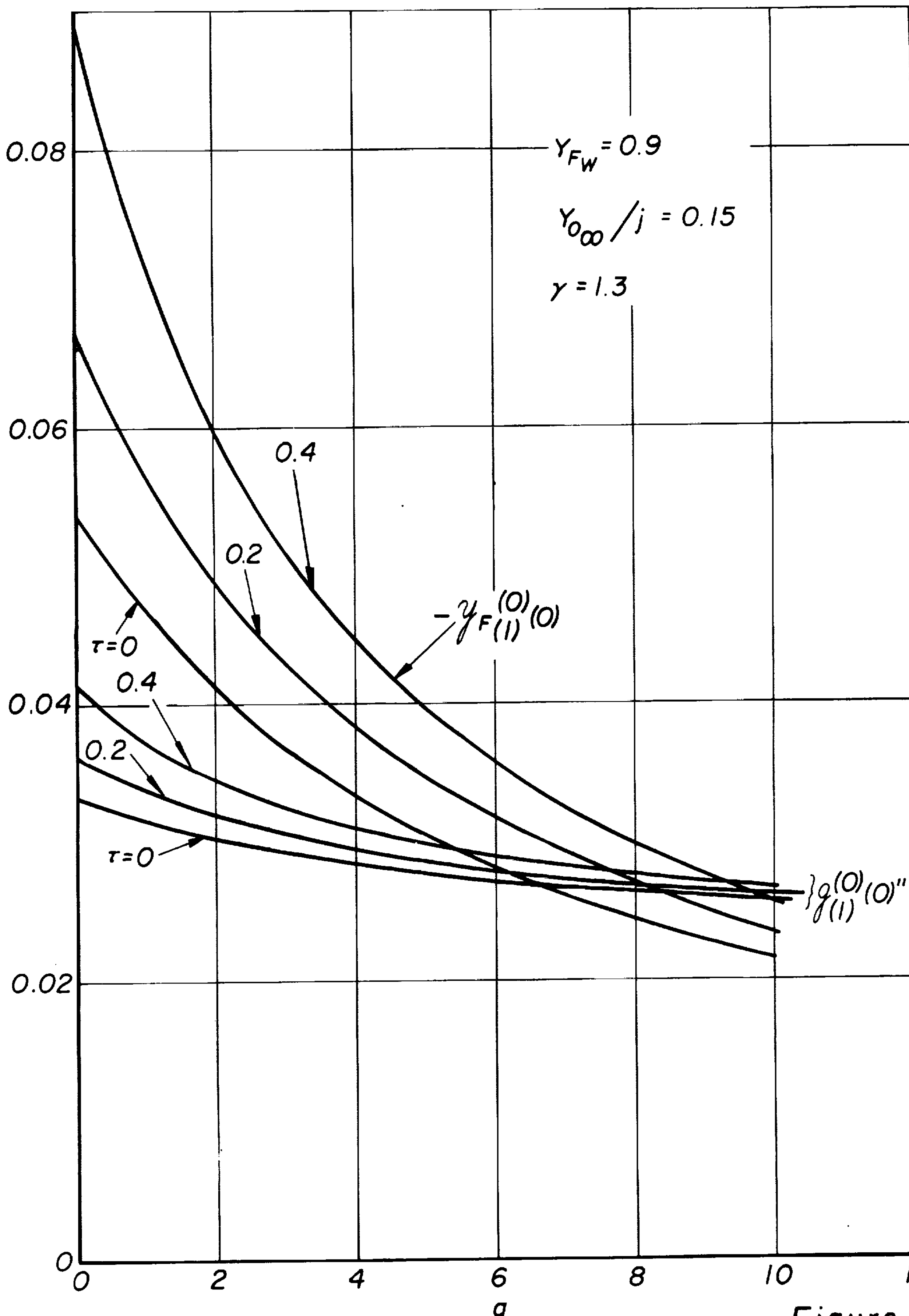


Figure 13

Quasi steady pressure response
of flat plate boundary layer

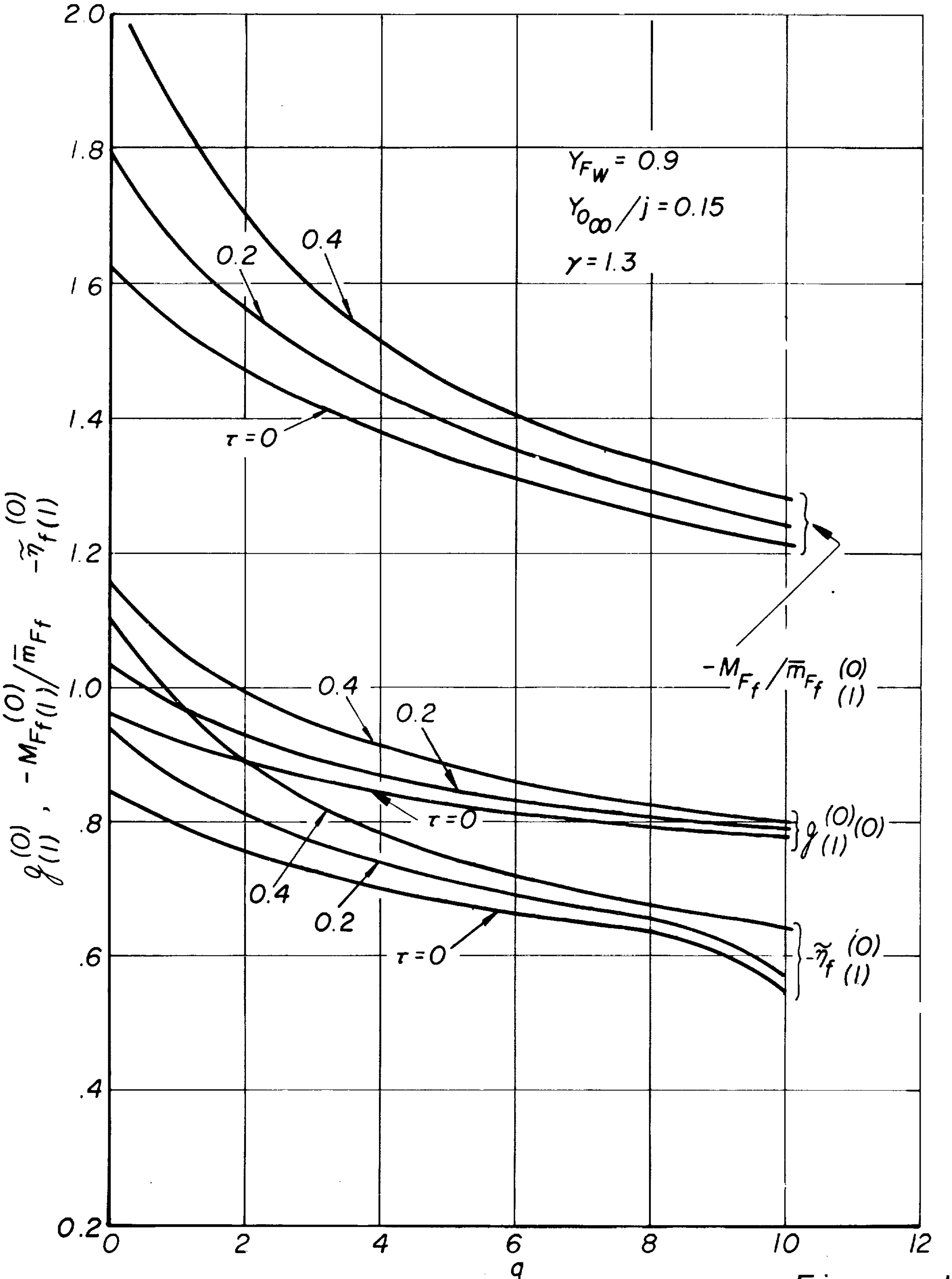


Figure 14

Velocity response of flat plate boundary layer

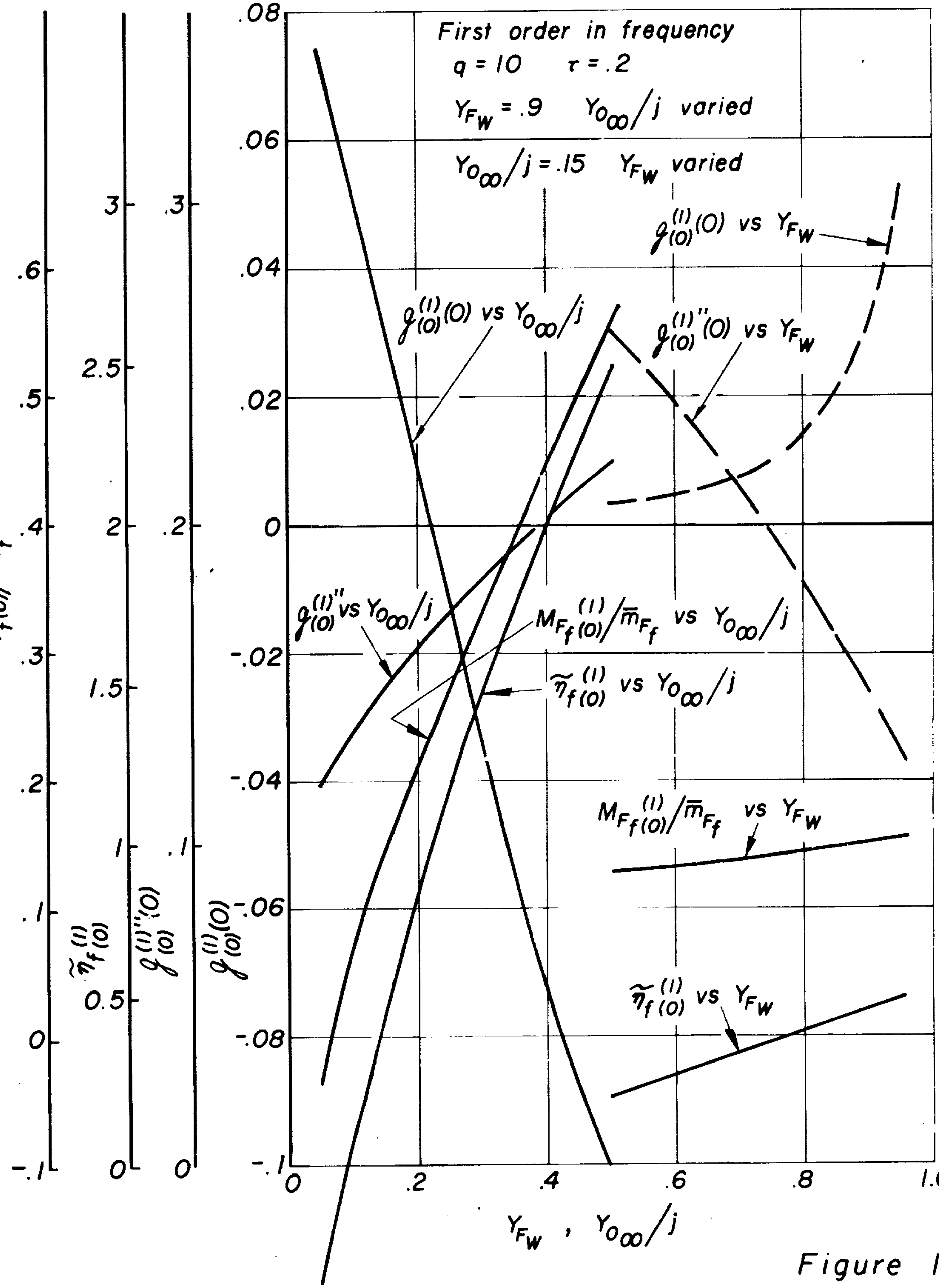


Figure 15

Velocity response of flat plate

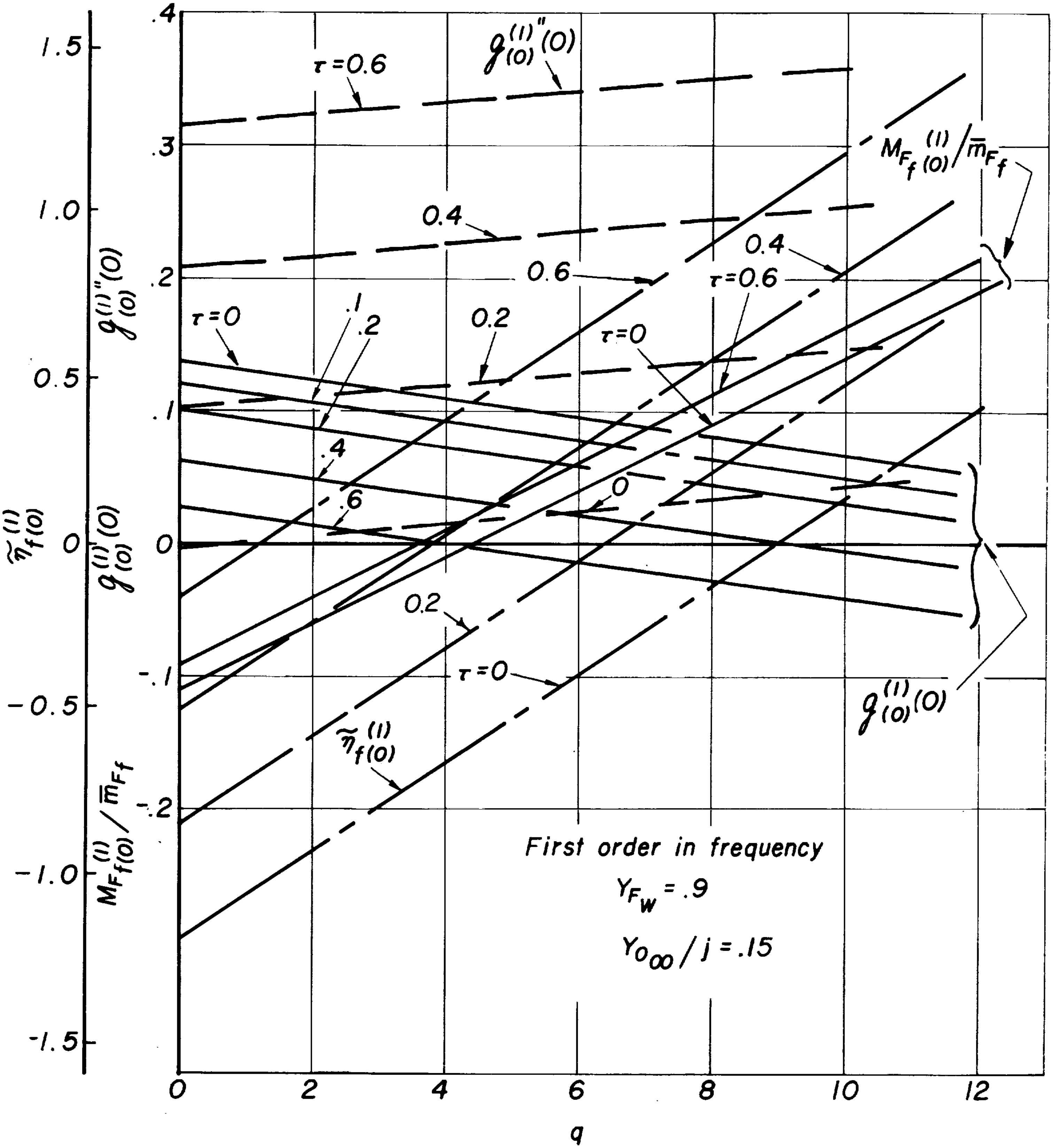


Figure 16

Stagnation point response pressure effect

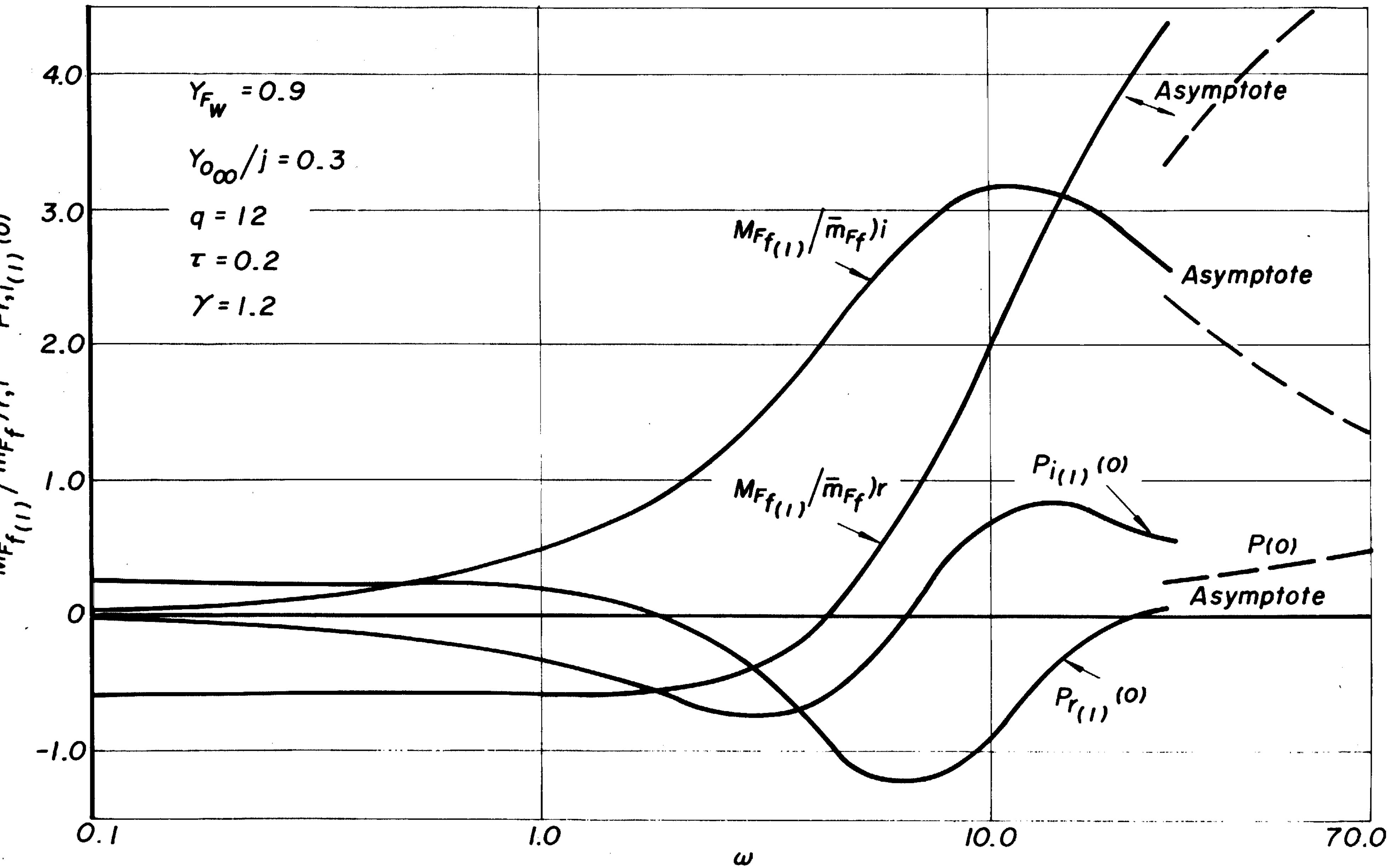


Figure 17a

Stagnation point response pressure effect

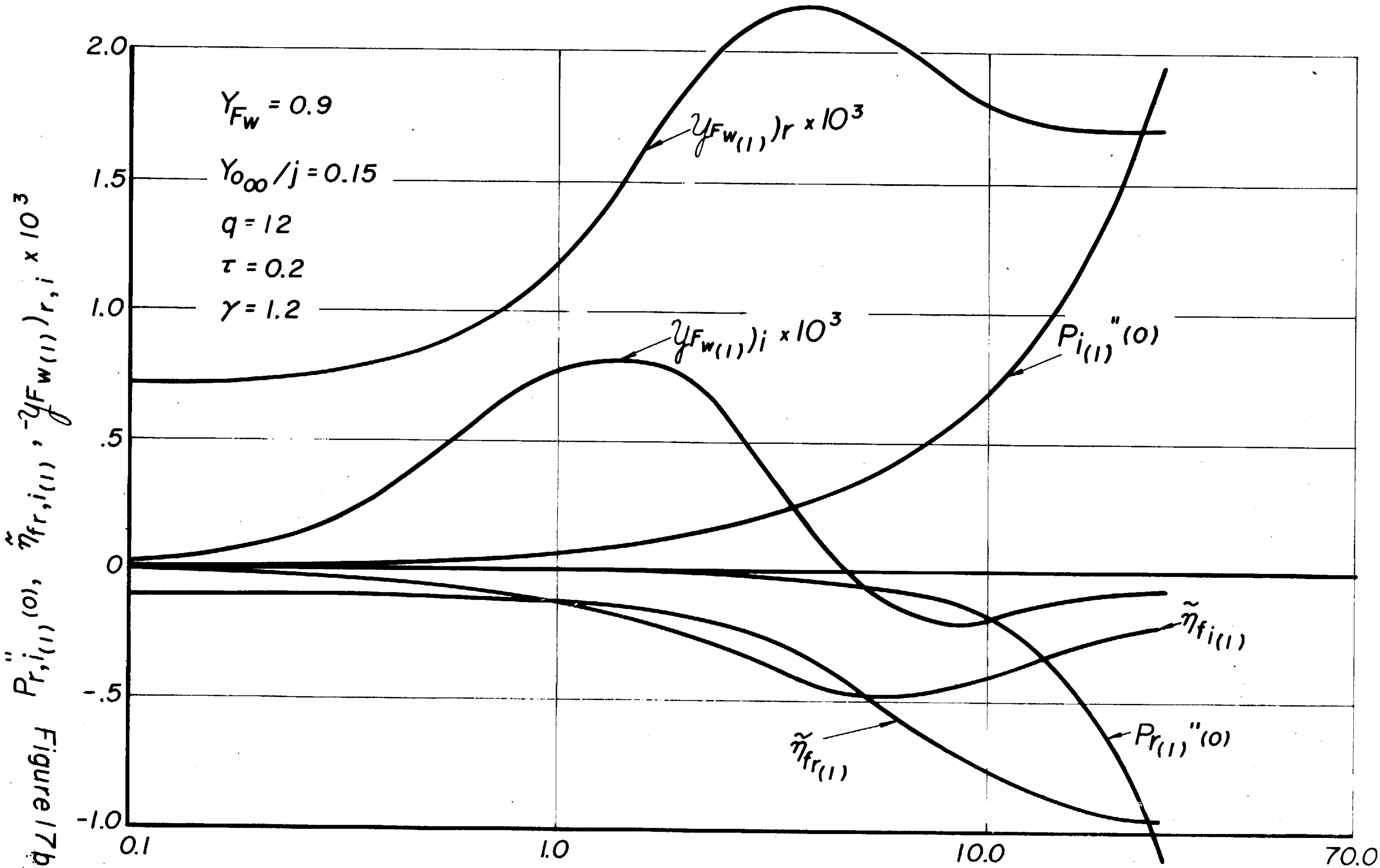


Figure 17b

Stagnation point response pressure effect

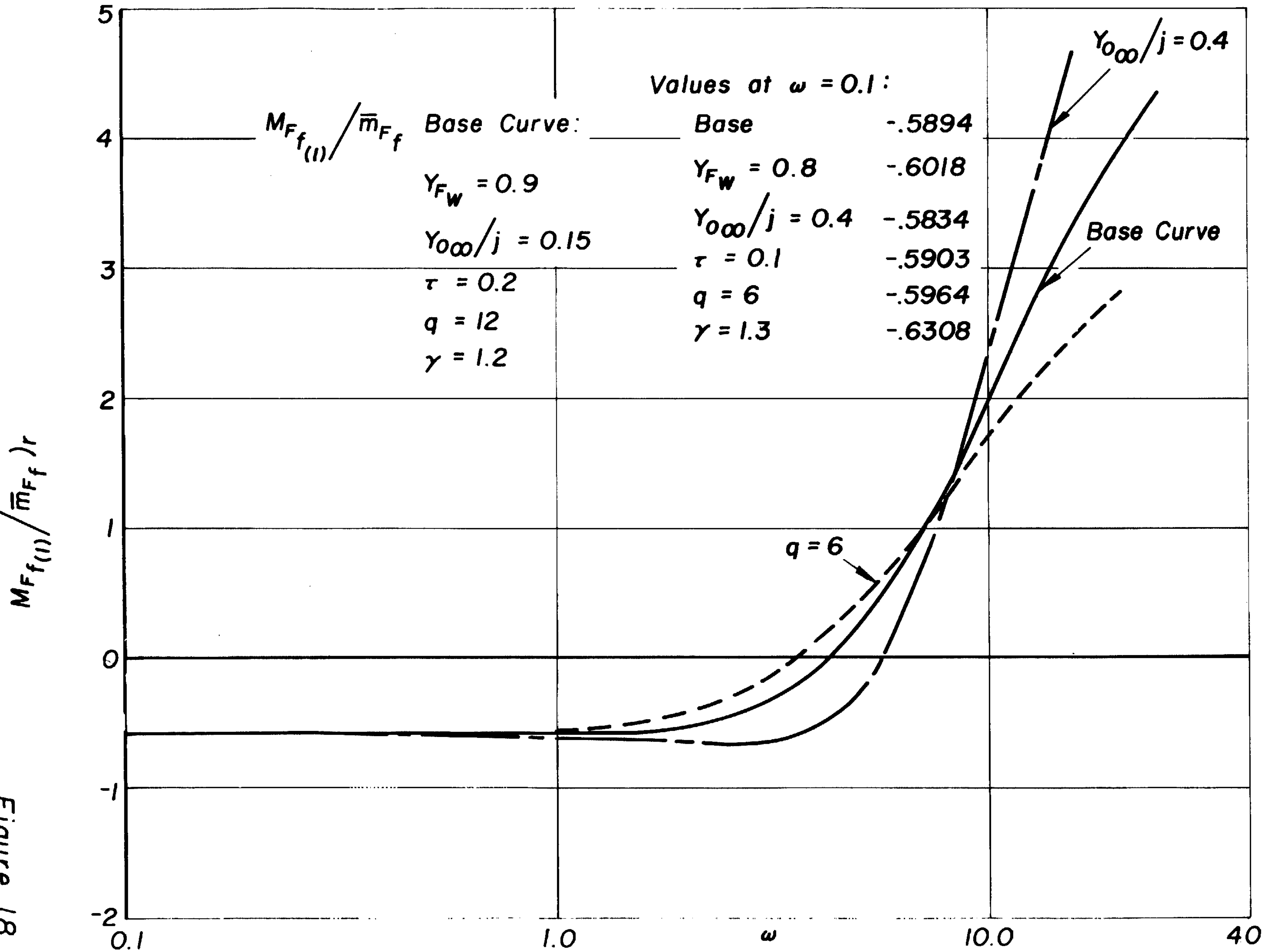


Figure 18

Stagnation point response pressure effect

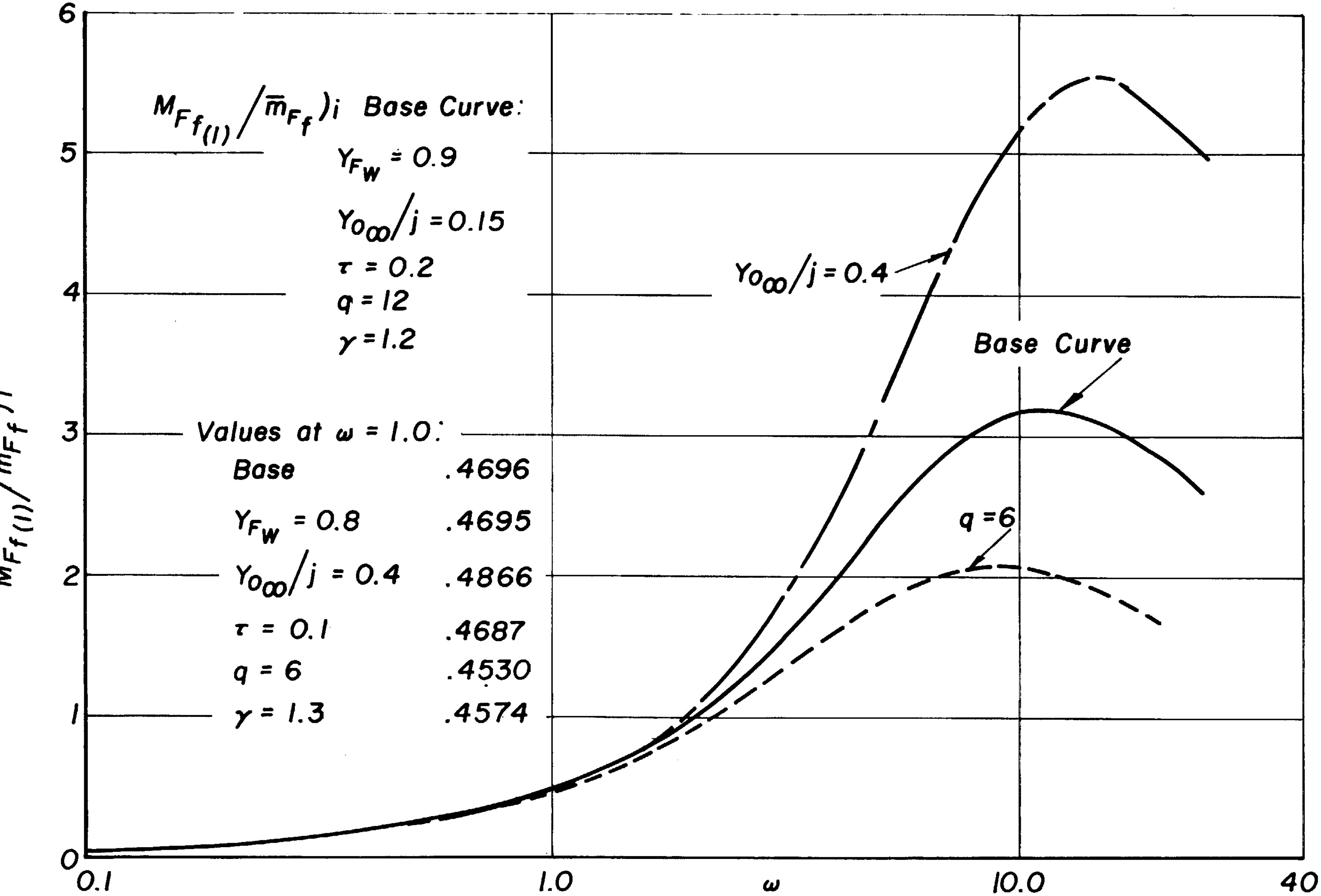


Figure 19

Stagnation point response pressure effect

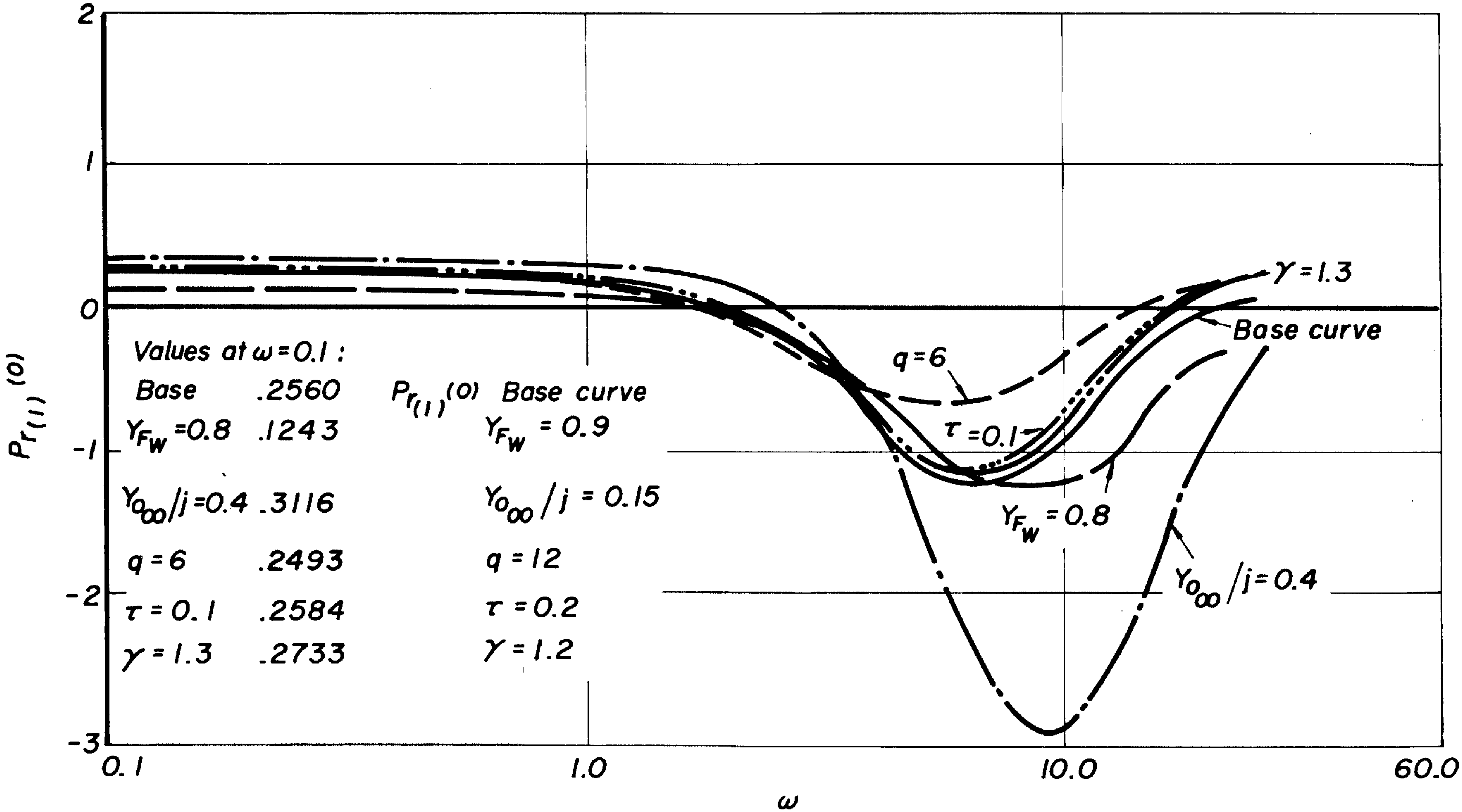
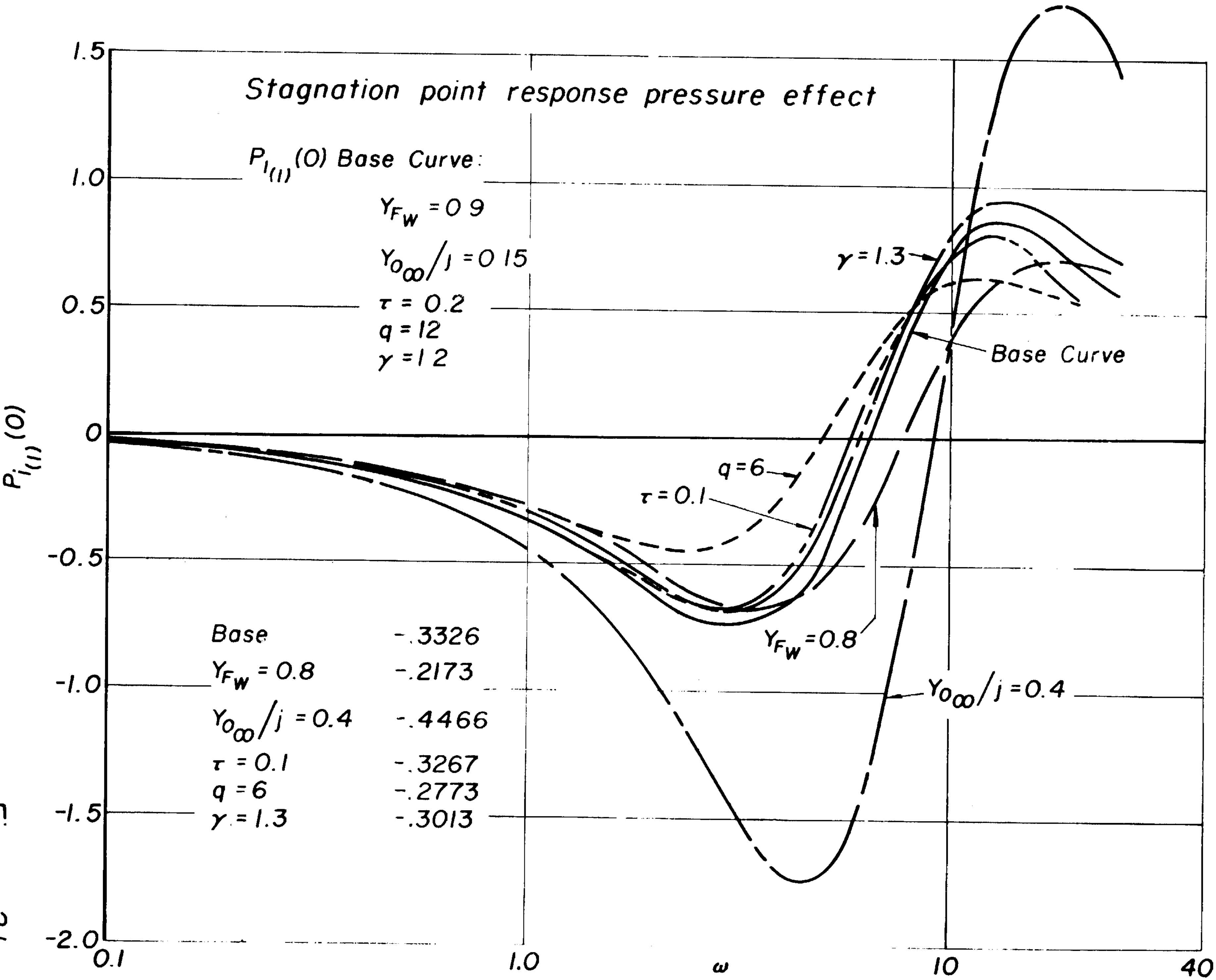
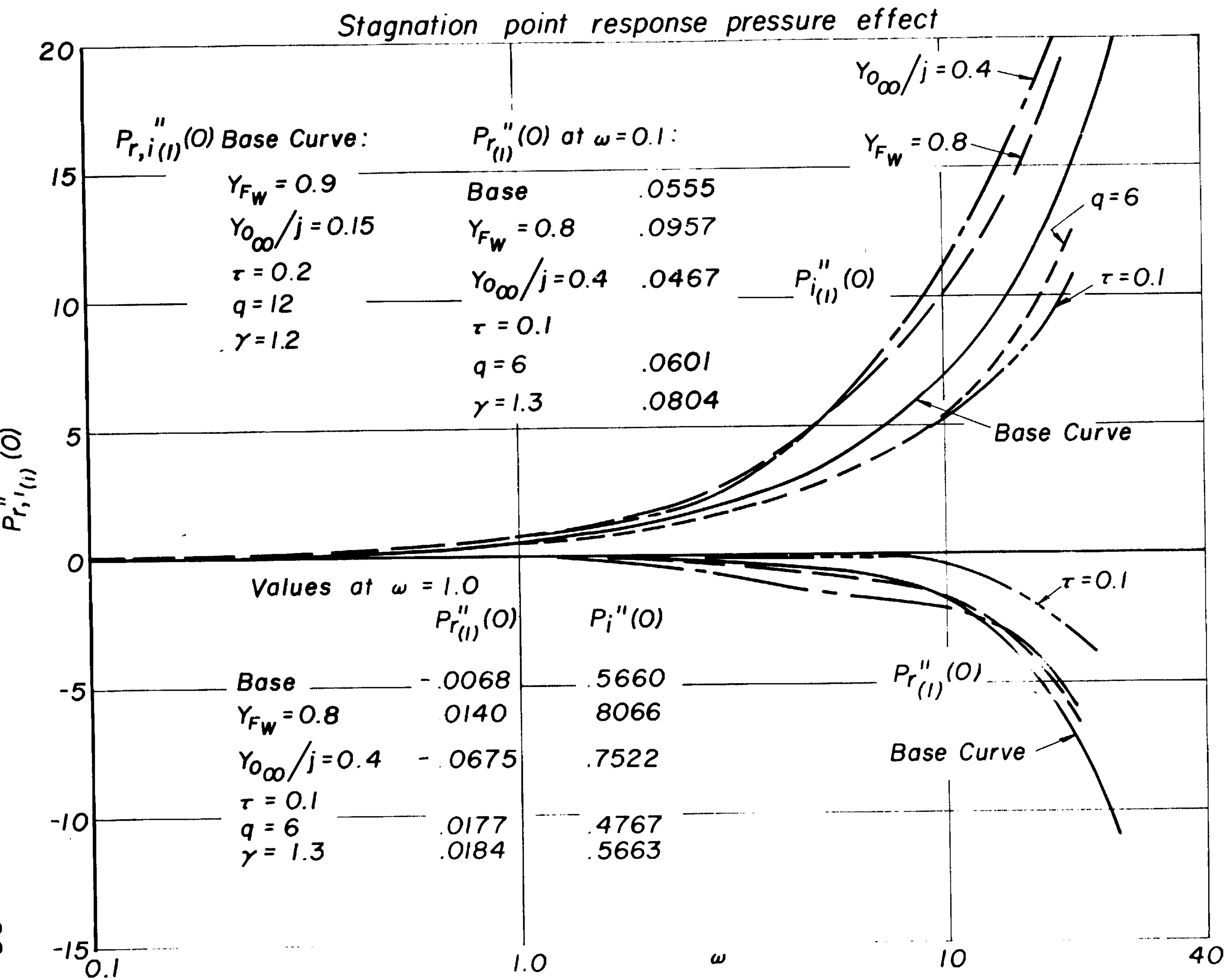


Figure 20



Stagnation point response pressure effect
$P_{1_{(1)}}(0)$ Base Curve:
$Y_{F_W} = 0.9$
$Y_{0_\infty}/j = 0.15$
$\tau = 0.2$
$q = 12$
$\gamma = 1.2$
Base -.3326
$Y_{F_W} = 0.8$ -.2173
$Y_{0_\infty}/j = 0.4$ -.4466
$\tau = 0.1$ -.3267
$q = 6$ -.2773
$\gamma = 1.3$ -.3013
$\gamma = 1.3$
Base Curve
$q = 6$
$\tau = 0.1$
$Y_{F_W} = 0.8$
$Y_{0_\infty}/j = 0.4$
$P_{1_{(1)}}(0)$
ω
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
-2.0
0.1
1.0
10
40

Figure 21



Stagnation point response pressure effect
$P''_{r,i(1)}(0)$ Base Curve:
$Y_{F_W} = 0.9$
$Y_{O_\infty}/j = 0.15$
$\tau = 0.2$
$q = 12$
$\gamma = 1.2$
$P''_{r(1)}(0)$ at $\omega = 0.1$:
Base .0555
$Y_{F_W} = 0.8$ .0957
$Y_{O_\infty}/j = 0.4$ .0467
$\tau = 0.1$
$q = 6$ .0601
$\gamma = 1.3$ .0804
Values at $\omega = 1.0$
$P''_{r(1)}(0)$ $P_i''(0)$
Base -.0068 .5660
$Y_{F_W} = 0.8$ .0140 8066
$Y_{O_\infty}/j = 0.4$ -.0675 .7522
$\tau = 0.1$
$q = 6$ .0177 .4767
$\gamma = 1.3$ .0184 .5663
$Y_{O_\infty}/j = 0.4$
$Y_{F_W} = 0.8$
$q = 6$
$\tau = 0.1$
$P''_{i(1)}(0)$
Base Curve
$\tau = 0.1$
$P''_{r(1)}(0)$
Base Curve
$P''_{r,i(1)}(0)$
20
15
10
5
0
-5
-10
-15
0.1
1.0
ω
10
40

Figure 22

Stagnation point response velocity effect

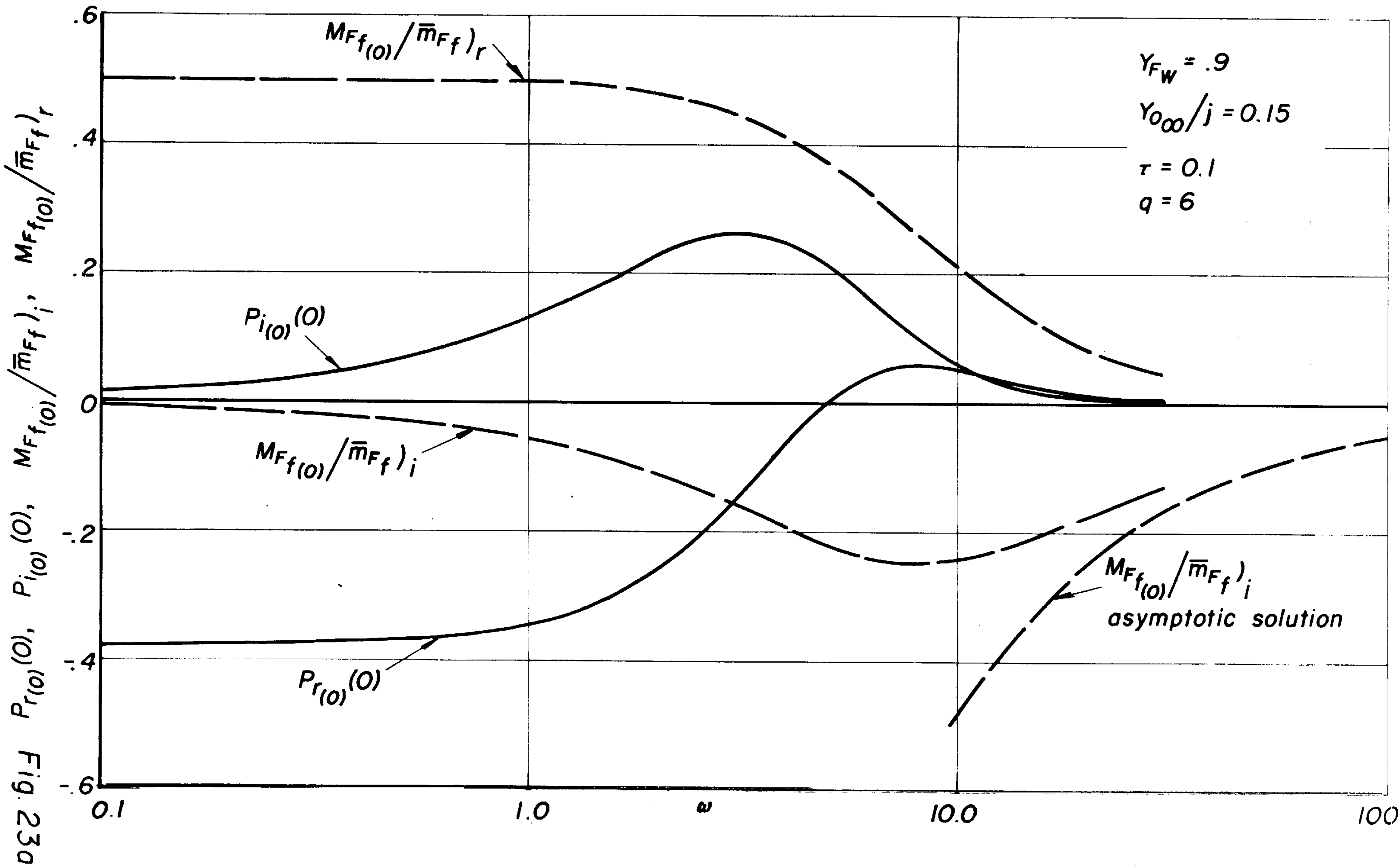


Fig. 23a

Stagnation point response velocity effect

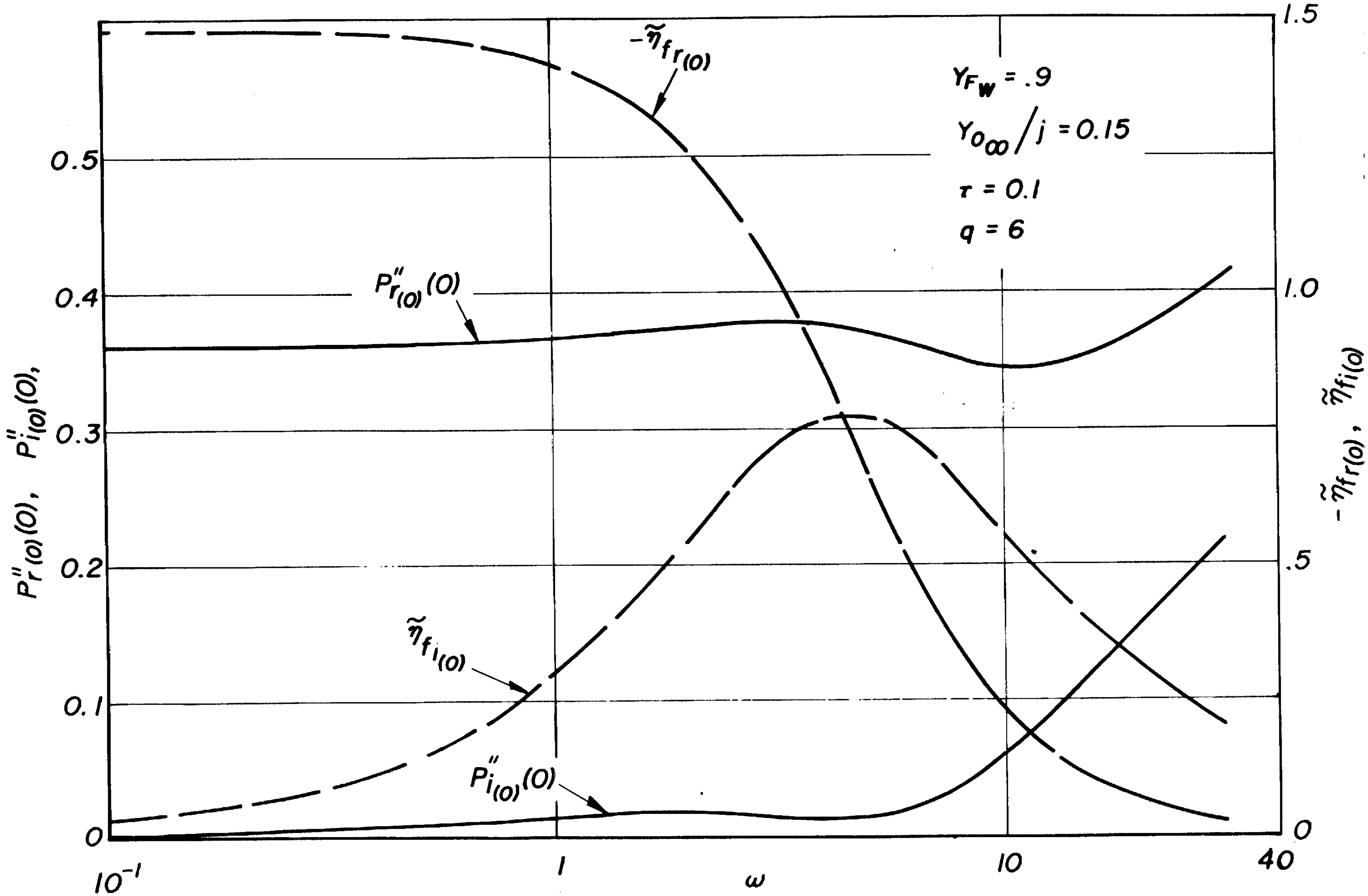
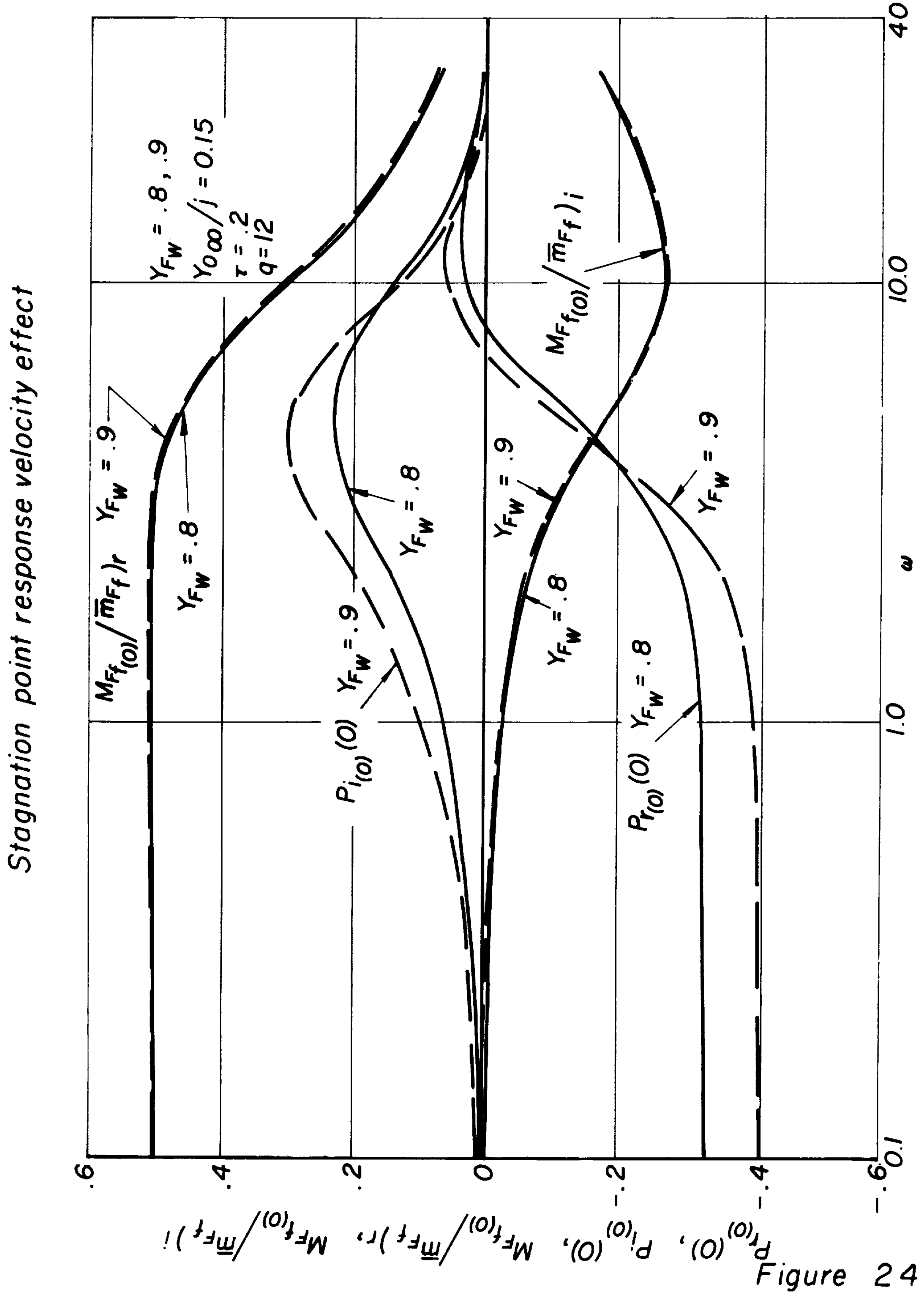


Figure 23b



Stagnation point response velocity effect
$M_{F_{f(0)}}/(\overline{m}_{F_f})_r$
$Y_{F_W} = .9$
$Y_{F_W} = .8$
$Y_{F_W} = .8, .9$
$Y_{0\infty}/j = 0.15$
$\tau = .2$
$q = 12$
$P_{i(0)}(0)$
$Y_{F_W} = .9$
$Y_{F_W} = .8$
$Y_{F_W} = .9$
$Y_{F_W} = .8$
$M_{F_{f(0)}}/(\overline{m}_{F_f})_i$
$P_{r(0)}(0)$
$Y_{F_W} = .8$
$Y_{F_W} = .9$
.6
.4
.2
0
-.2
-.4
-.6
0.1
1.0
10.0
40
ω
$P_{r(0)}(0),\ P_{i(0)}(0),\ M_{F_{f(0)}}/(\overline{m}_{F_f})_r,\ M_{F_{f(0)}}/(\overline{m}_{F_f})_i$

Figure 24a

Stagnation point response velocity effect

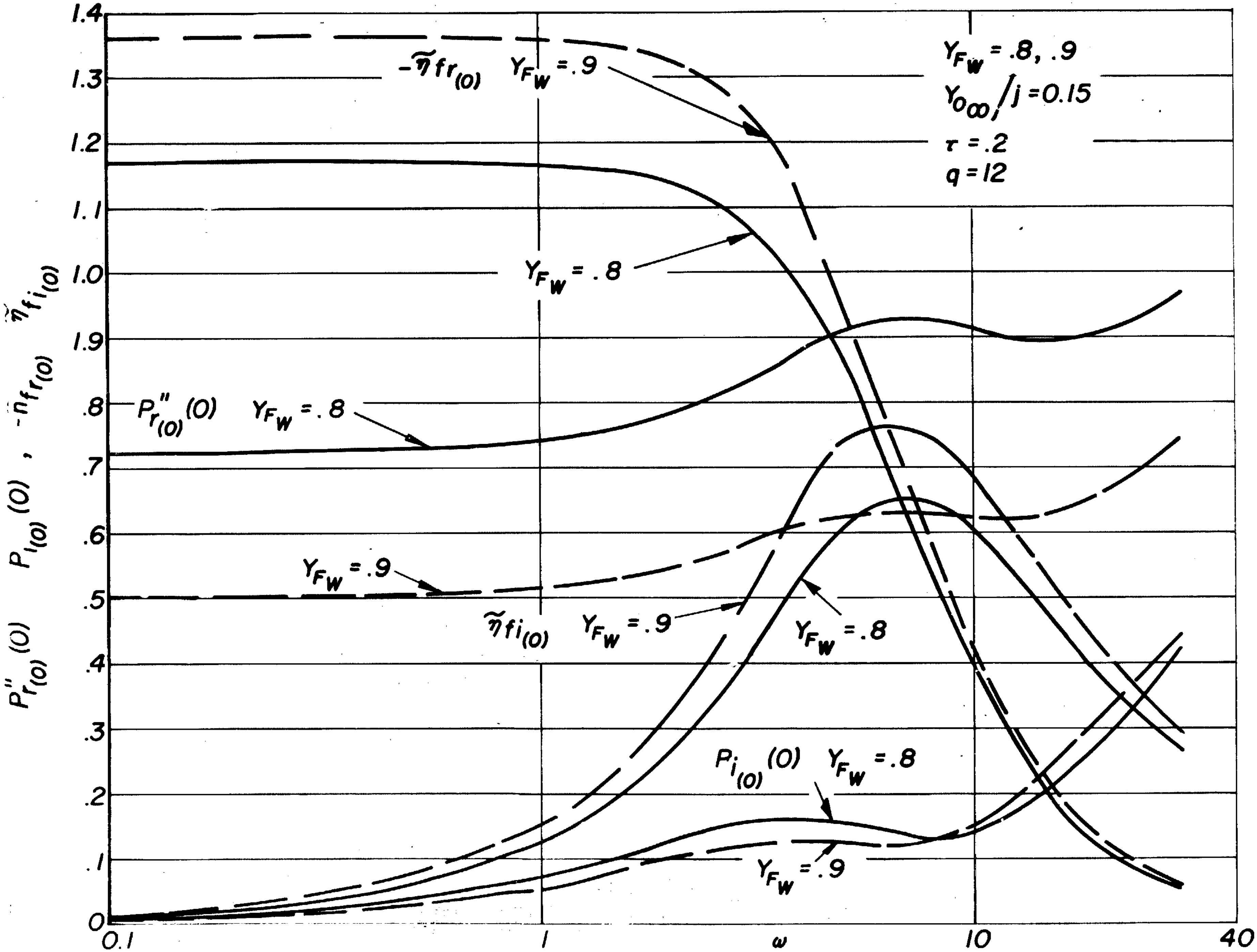


Figure 24b

Stagnation point response velocity effect

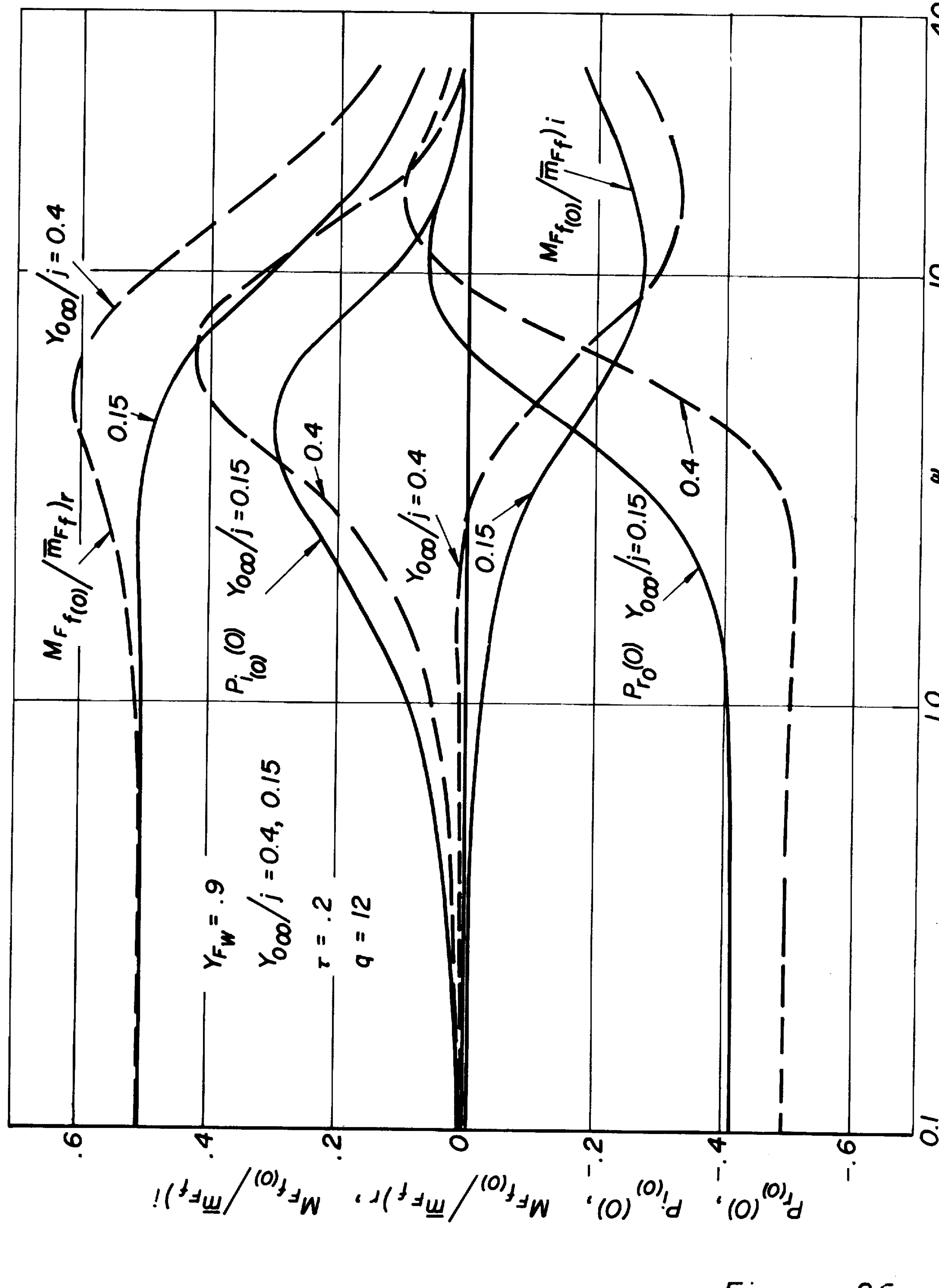


Figure 25a

Stagnation point response velocity effect

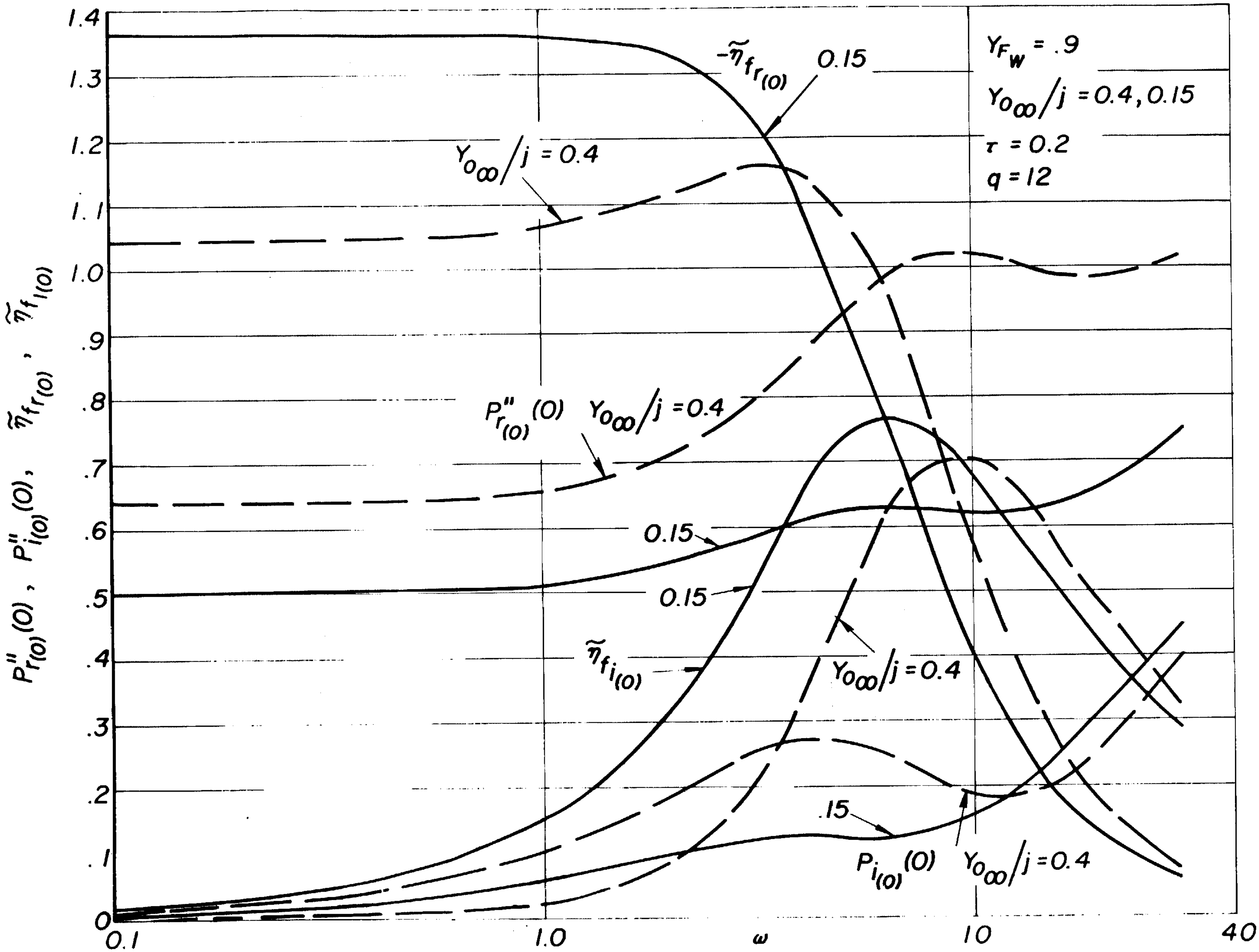


Figure 25b

Stagnation point response velocity effect

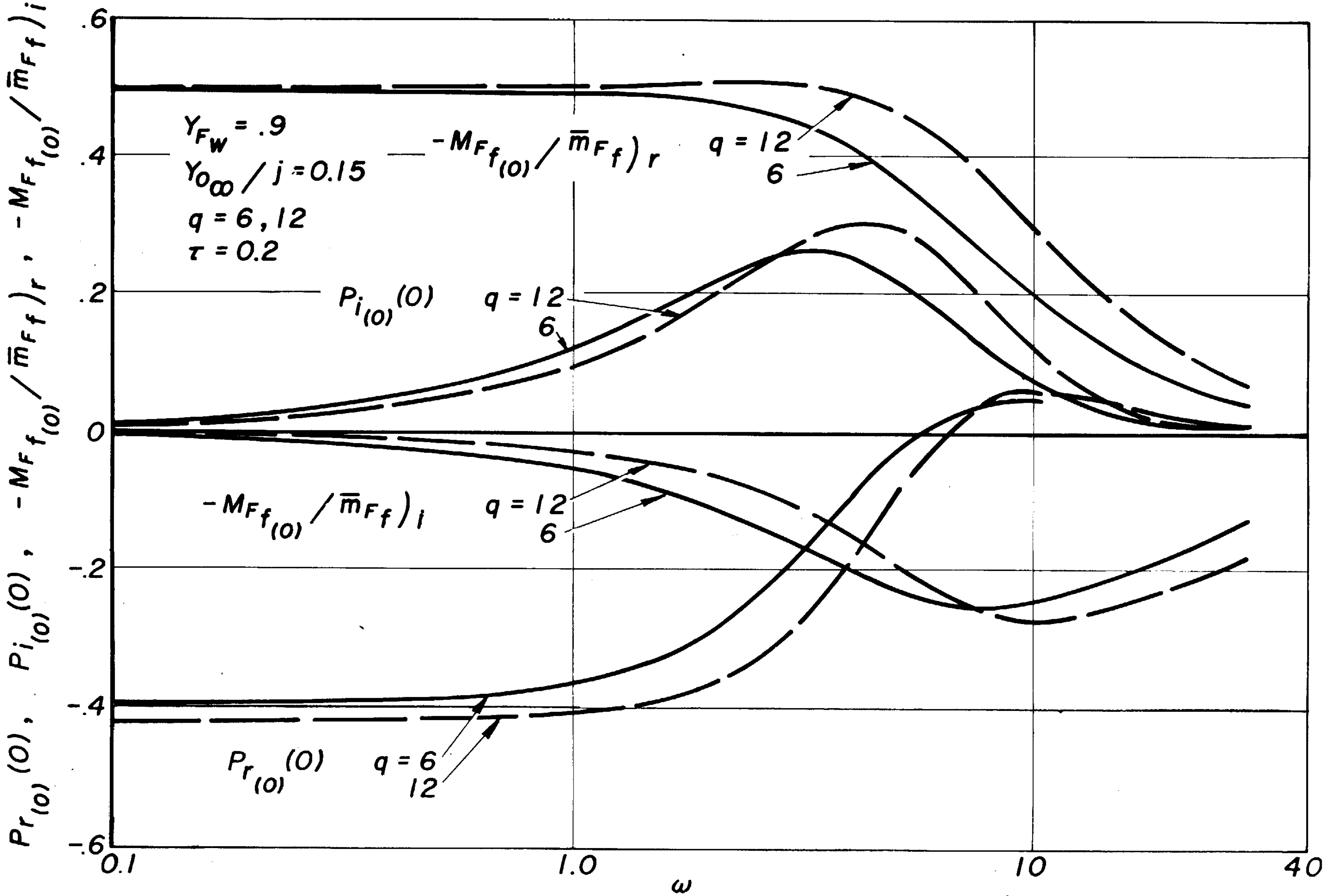


Figure 26a

Stagnation point response velocity effect

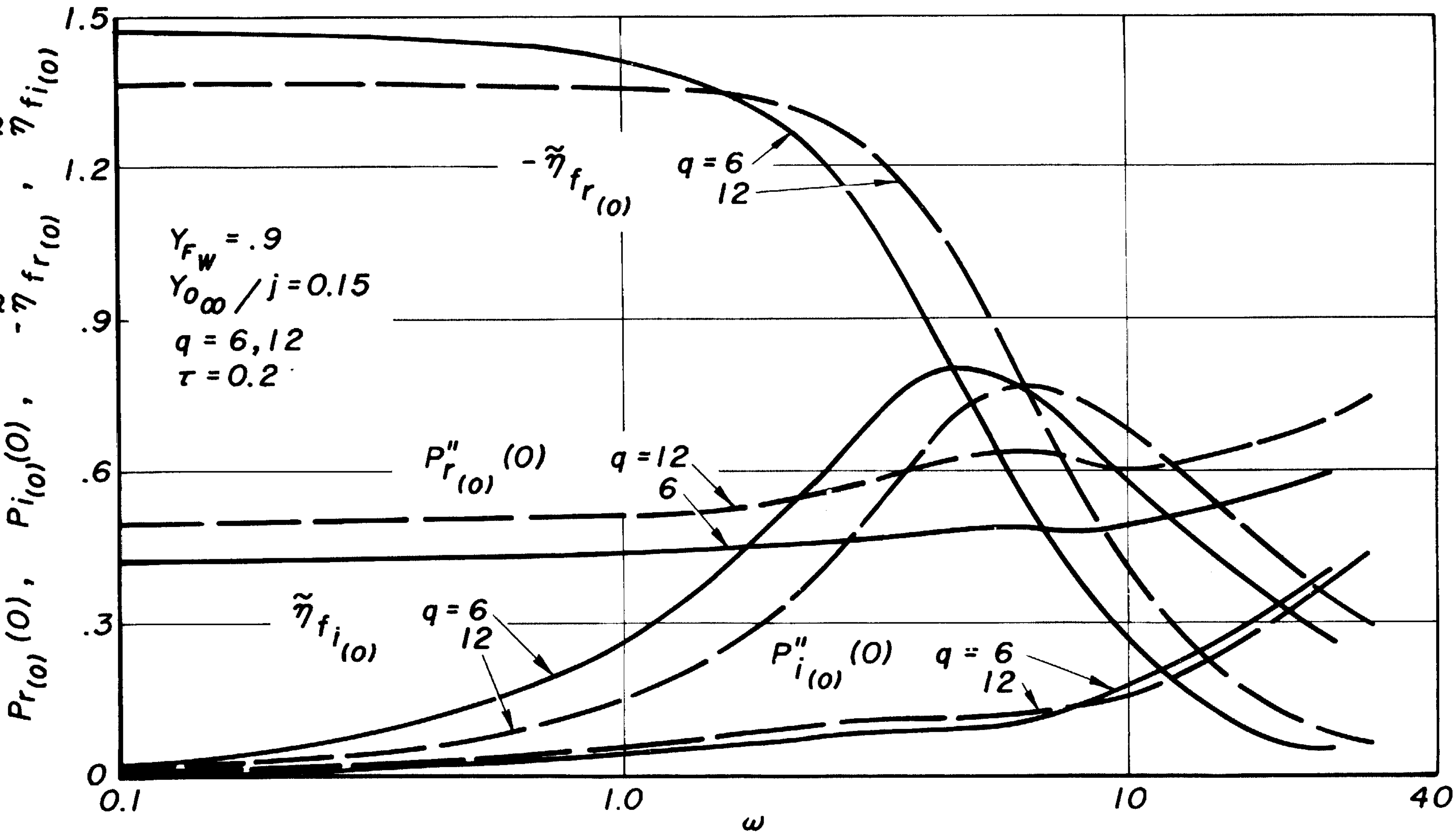


Figure 26b

Stagnation point response velocity effect

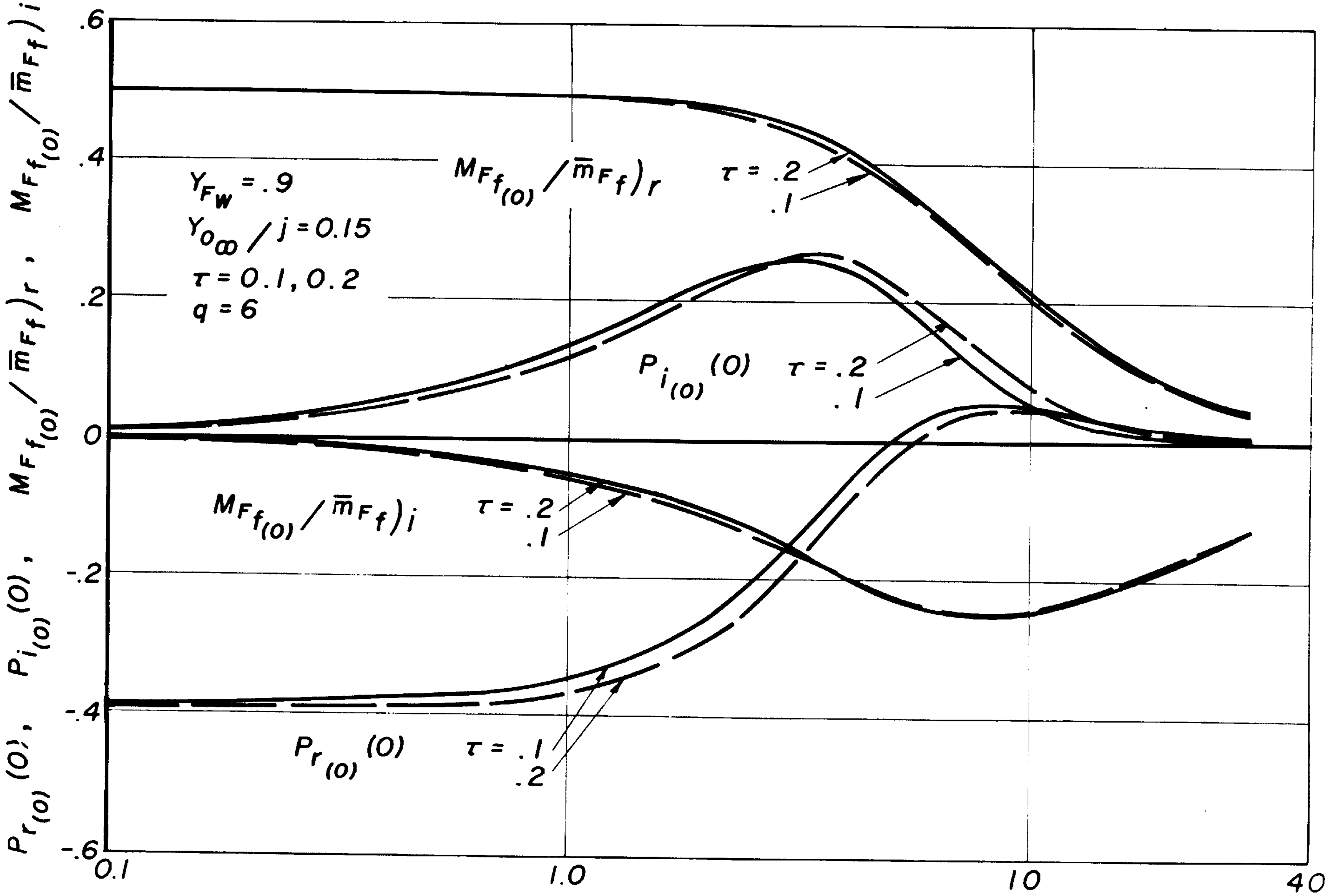


Figure 27a

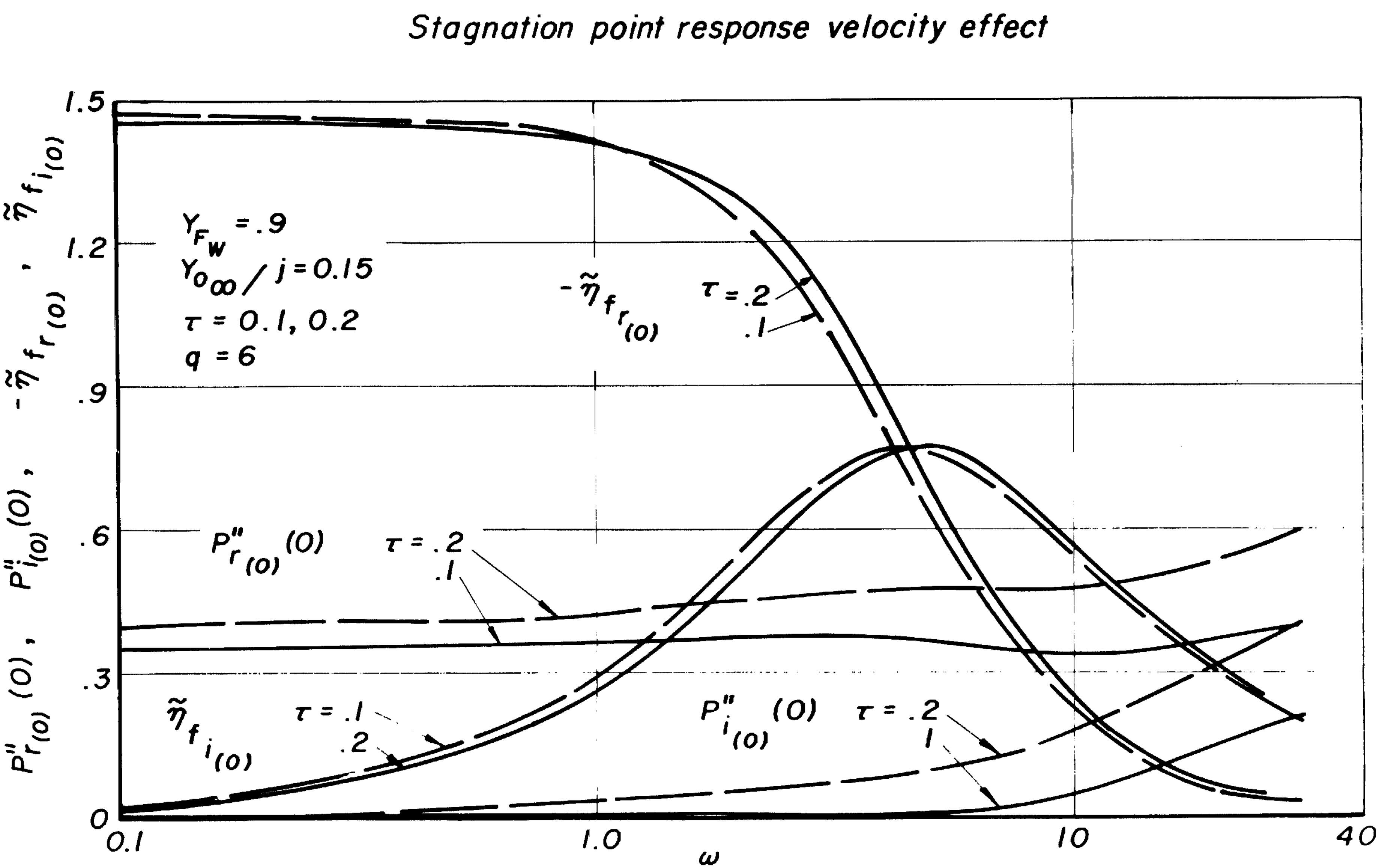

Stagnation point response velocity effect
$Y_{F_w} = .9$
$Y_{O_\infty}/j = 0.15$
$\tau = 0.1, 0.2$
$q = 6$
$-\tilde{\eta}_{f_{r(0)}}$
$\tau = .2$
$.1$
$P''_{r(0)}(0)$
$\tau = .2$
$.1$
$\tilde{\eta}_{f_{i(0)}}$
$\tau = .1$
$.2$
$P''_{i(0)}(0)$
$\tau = .2$
$.1$
1.5
1.2
.9
.6
.3
0
0.1
1.0
10
40
ω
$P''_{r(0)}(0)$, $P''_{i(0)}(0)$, $-\tilde{\eta}_{f_{r(0)}}$, $\tilde{\eta}_{f_{i(0)}}$

Figure 27b

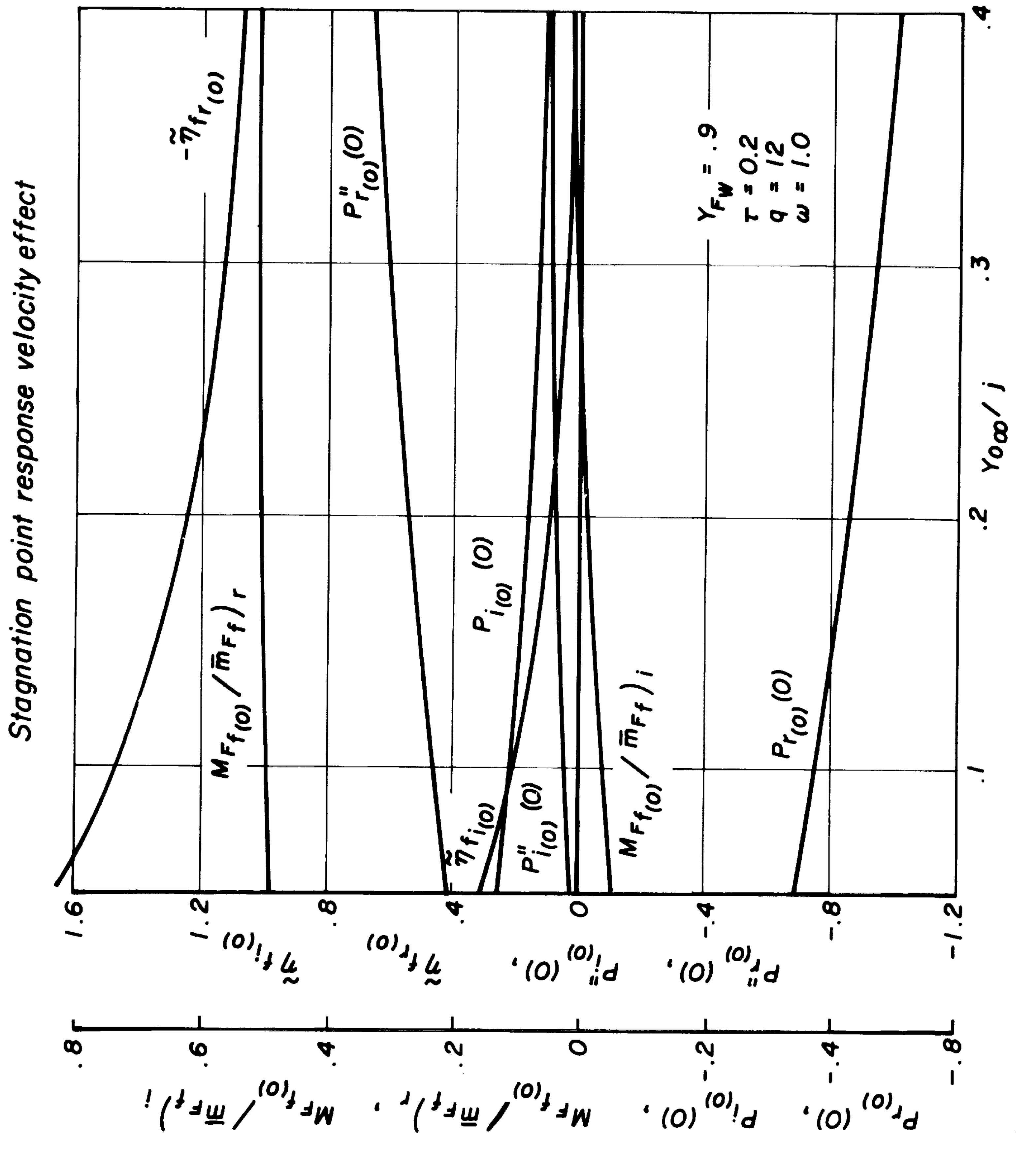

Stagnation point response velocity effect
$-\tilde{\eta}_{f_{r(0)}}$
$P''_{r(0)}(0)$
$M_{F_{f(0)}}/(\bar{m}_{F_f})_r$
$P_{i(0)}(0)$
$\tilde{\eta}_{f_{i(0)}}$
$P''_{i(0)}(0)$
$M_{F_{f(0)}}/(\bar{m}_{F_f})_i$
$P_{r(0)}(0)$
$Y_{F_W} = .9$
$\tau = 0.2$
$q = 12$
$\omega = 1.0$
$Y_{O\infty}/j$
$\tilde{\eta}_{f_{i(0)}}$, $\tilde{\eta}_{f_{r(0)}}$, $P''_{i(0)}(0)$, $P''_{r(0)}(0)$
$P_{r(0)}(0)$, $P_{i(0)}(0)$, $M_{F_{f(0)}}/(\bar{m}_{F_f})_r$, $M_{F_{f(0)}}/(\bar{m}_{F_f})_i$

Figure 28

Stagnation point response velocity effect

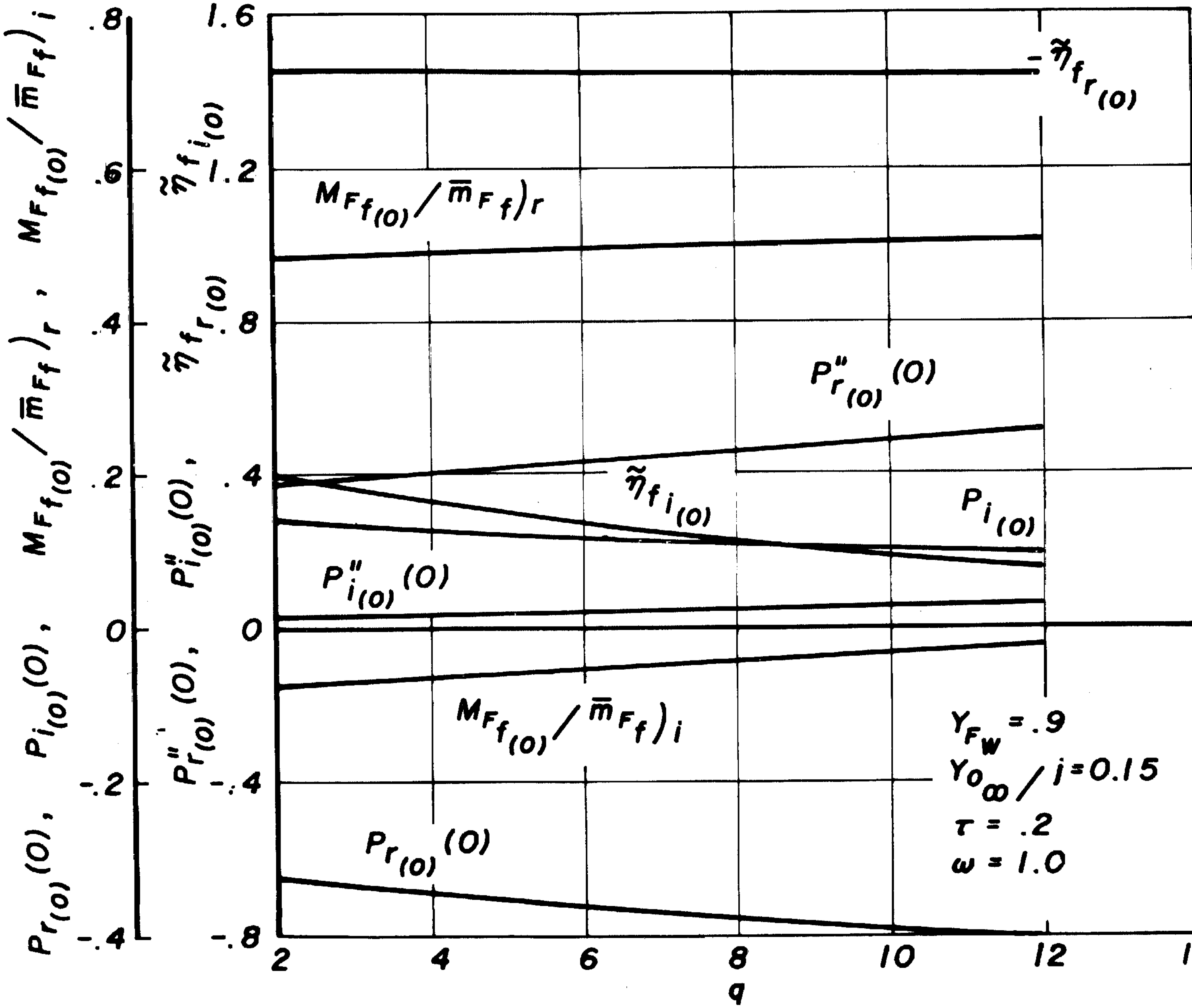


Figure 29

Conjecture of flat plate response velocity effect

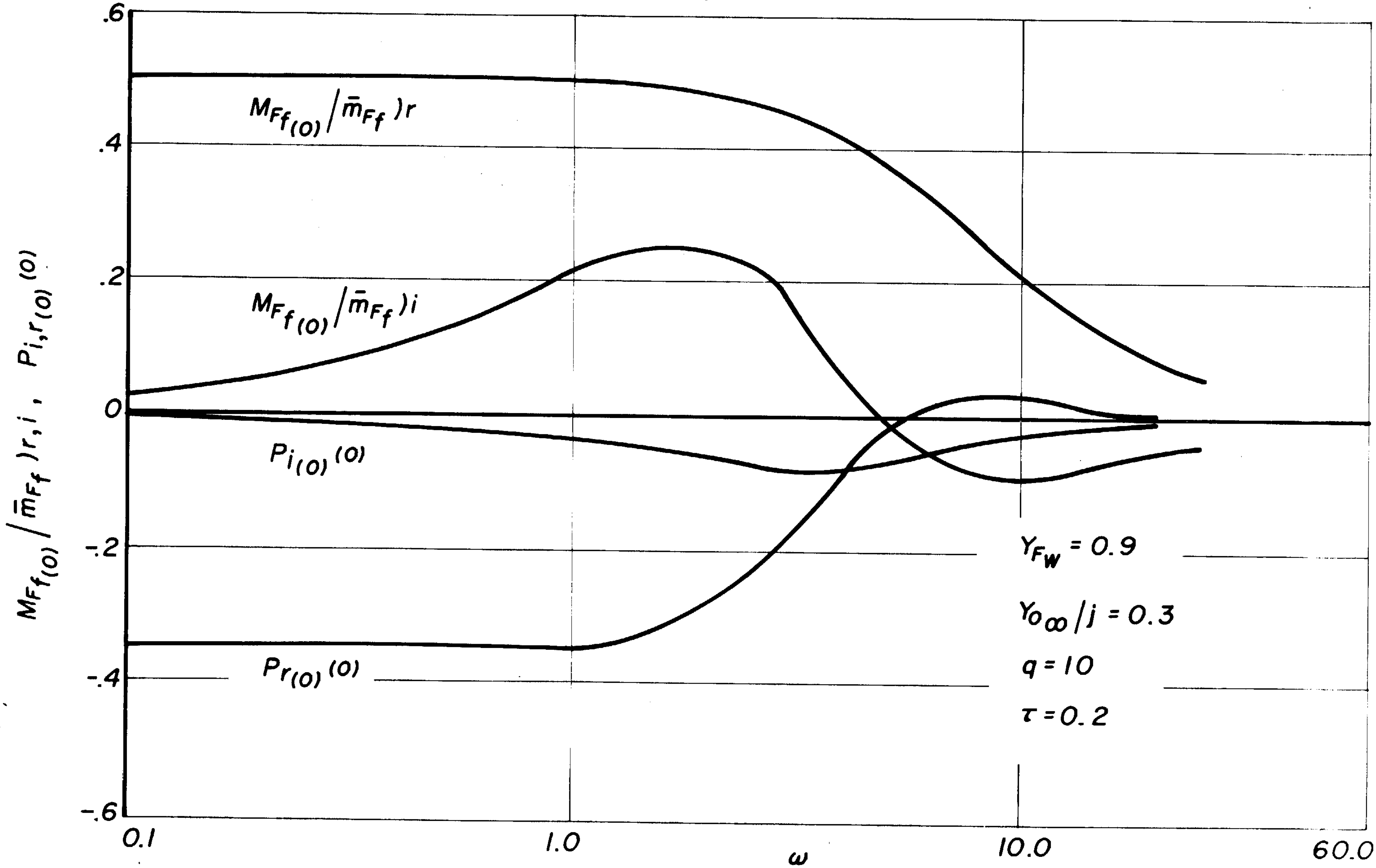


Figure 30a

Conjecture of flat plate response
pressure effect

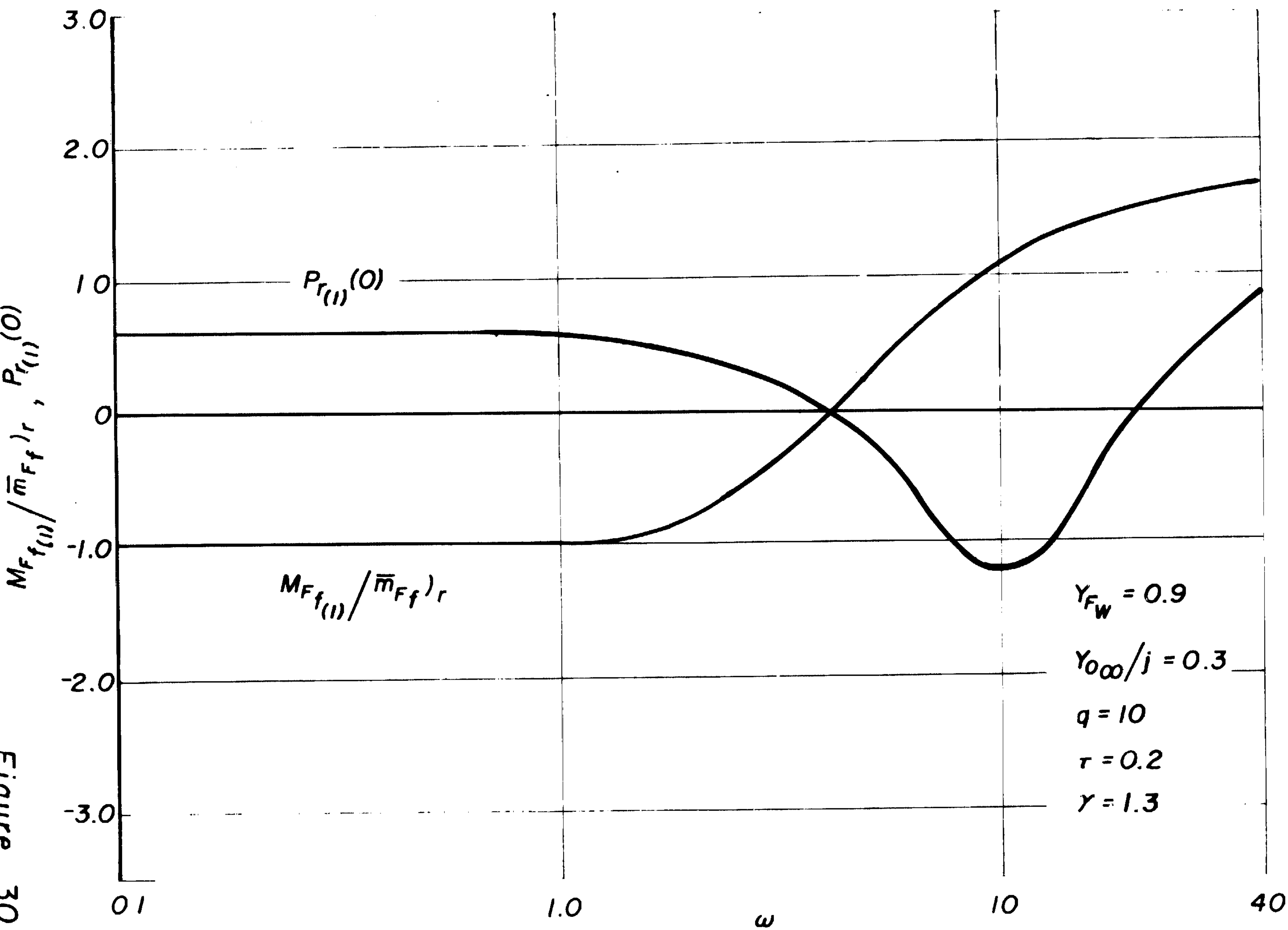


Figure 30b

APPENDIX A: THE CHARACTERISTIC TIMES OF DROPLET HEAT-UP, INITIAL CONDITION RELAXATION, AND UNSTEADY VAPORIZATION

Under the assumptions listed in Chapter II an energy balance at the droplet during the heat-up period yields

$$\frac{4}{3}\pi r_L^{*3}\rho_L^* c_L^* \frac{dT_L^*}{dt^*} = 4\pi r_L^{*2}\bar{\lambda}^* \left.\frac{dT^*}{dr^*}\right)_{r=r_L^*} \tag{A-1}$$

The gas temperature gradient at the droplet surface is evaluated from a solution to Eq. (1.21d) for the temperature distribution for no mass transfer (Laplace's equation). This is

$$\left.\frac{dT^*}{dr^*}\right)_{r=r_L^*} = \frac{T_f^* - \tau^*}{C\, r_L^*} \tag{A-2}$$

Combining Eqs. (A-1) and (A-2)

$$\frac{dT_L^*}{T_L^* - T_f^*} = \frac{3\bar{\lambda}^* \quad dt^*}{C\, r_L^{*2}\rho_L^* c_L^*}$$

Integrating,

$$\ln\left[\frac{T_L^* - T_f^*}{T_L^*(0) - T_f^*}\right] = \frac{3\bar{\lambda}^*}{C\, r_L^{*2}\rho_L^* c_L^*}\, t^*$$

For high temperature vaporization of volatile liquids the wet bulb temperature is quite small compared to T_f^*. Therefore the logarithm may be expanded and t^* evaluated when the wet bulb temperature is reached, T_{WB}^*. The heat-up

time is therefore

$$\tau_{HL}^{*} = \left(\frac{\rho_L^{*} c_L^{*} r_L^{*2}}{3\lambda^{*}} \right) C \left[\frac{T_{WB}^{*} - T_L(0)^{*}}{T_f^{*} - T_{WB}^{*}} + O\left[\left(\frac{T_{WB}^{*}}{T_f^{*}}\right)^2\right] \right] \tag{A-3}$$

Consider now the problems of surface contraction and initial conditions relaxation. Repeating the governing system of equations in Chapter II Eq's. (2.1a)-(2.4),

$$\frac{1}{K}\rho_t + \frac{1}{r^2}\hat{m}_r = 0 \tag{A-4a}$$

$$\frac{1}{K}T_t + vT_r = \frac{1}{\rho r^2}(r^2 T_r)_r \tag{A-4b}$$

$$\rho T = 1 \tag{A-4c}$$

$$T(r, 0) = T_0(r) \tag{A-5}$$

$$T[r_L(t), t] = \tau \quad ; \quad T[r_f(t), t] = 1 \tag{A-6}$$

$$T_r\Big)_{r_L(t)} = \frac{\hat{m}[r_L(t)]}{r_L^2 B_\infty} = -\frac{1}{K_1 B_\infty}\frac{dr_L}{dt} \tag{A-7}$$

An exact solution of the system of equations in series form is first attempted. First, from Eq. (A-4b), using Eq's. (A-4a) and (A-4c),

$$(r^2 T_r - \hat{m}T)_r = 0 \tag{A-8}$$

This is immediately integrable over r ; applying Eq. (A-7),

$$r^2 T_r - \hat{m}T = m_L(t)\left(\frac{1}{B_\infty} - \tau\right) \tag{A-9}$$

where $\hat{m}_L(t)$ is the mass flow rate at the liquid surface which is, of course, a function of time. It may seem fruitful to obtain a single equation in T alone by the elimination of $\hat{m}$ from Eq's. (A-4a) and (A-9). However, such a procedure requires the use of a more elaborate expansion procedure than that which may be used if the full set of equations is retained. First note the appearance of four fundamental parameters in Eq's. (A-4a), (A-5), (A-7), and (A-9); these are κ, A, τ, and κ_1, where B_∞ may be derived from A and τ. Another useful parameter which may be derived is the Spalding transfer number:

$$B = B_\infty(1-\tau) = \frac{1-\tau}{\tau+A}$$

As has been previously mentioned $\frac{1}{\kappa} \ll 1$ in a great many problems of interest. Therefore, a Taylor series expansion of the solution in $1/\kappa$ is assumed valid:

$$T(r,t) = \sum_{i=0}^{\infty} \left(\frac{1}{\kappa}\right)^i T^{(i)}$$

$$r_L(t) = \sum_{i=0}^{\infty} \left(\frac{1}{\kappa}\right)^i r_L^{(i)}$$

A similar expansion is assumed to hold for the other variables of interest $\hat{m}$, ρ, etc. Substituting the expansion into the governing equations, the following system is obtained for the zeroth order in $1/\kappa$:

$$r^2 T_r^{(0)} - \hat{m}^{(0)} T^{(0)} = \hat{m}_L^{(0)} A \qquad \text{(A-10a)}$$

$$\hat{m}_r^{(0)} = 0 \tag{A-10b}$$

$$\rho^{(0)} T^{(0)} = 1 \tag{A-10c}$$

$$T^{(0)}[r_L^{(0)}(t), t] = \tau \tag{A-10d}$$

$$T^{(0)}[r_f^{(0)}(t), t] = 1 \tag{A-10e}$$

$$T^{(0)}(r, 0) = T_0(r) \tag{A-10f}$$

$$T_r^{(0)}\Big)_{r_L^{(0)}} = \frac{\hat{m}_L^{(0)}}{B_\infty r_L^{(0)2}} = -\frac{1}{B_\infty K_1} \frac{d r_L^{(0)}}{dt} \tag{A-10g}$$

The first order system is

$$r^2 T_r^{(1)} - \hat{m}^{(1)} T^{(0)} - \hat{m}^{(0)} T^{(1)} = \hat{m}_L^{(1)} A \tag{A-11a}$$

$$\hat{m}_r^{(1)} = -r^2 \rho_t^{(0)} \tag{A-11b}$$

$$\rho^{(0)} T^{(1)} + \rho^{(1)} T^{(0)} = 0 \tag{A-11c}$$

$$T^{(1)}[r_L^{(0)}(t), t] + T_r^{(0)}[r_L^{(0)}(t), t] \; r_L^{(1)} = 0 \tag{A-11d}$$

$$T^{(1)}(r_f^{(0)}) + T_r^{(0)}(r_f^{(0)}) \; r_f^{(1)} = 0 \tag{A-11e}$$

$$T^{(1)}(r, 0) = 0 \tag{A-11f}$$

(A-11g)

An important observation is immediately apparent. The initial conditions, Eq. (A-5) or Eq's. (A-10f) and (A-11f), can never be satisfied with this scheme since, as in the quasi-steady solution, the time derivative of the order of the solution being considered never appears in the equation. This could have been seen at the outset and is analogous to problems that arise in, say, ordinary differential equations when a regular expansion in terms of a small parameter appearing in front of the highest derivative is attempted. The usual procedure in such a case is to find a scale transformation of variable. While this procedure could have been adopted here, no transformation yielding equations amenable to exact analysis has been found. Now, although the idea of an exact solution has been abandoned, this solution should yield information concerning one of the two types of unsteadiness which enters the problem. Assuming that the initial conditions which will be demanded by this solution can be provided, information should be gained concerning the unsteady effects introduced by the contracting droplet radius. In particular, under the imposed conditions, a correction to the quasi-steady vaporization rate should be obtained.

Proceeding on this basis, Eq's. (A-10a) and (A-10b) are merely the quasi-steady equations. They may be readily integrated to yield

$$T^{(0)} + A = \frac{e^{\hat{m}^{(0)}\left(\frac{1}{r_L^{(0)}} - \frac{1}{r}\right)}}{B_\infty} \tag{A-12}$$

Using Eq's. (A-10c) and (A-10g), the familiar D^2 law may be obtained;

$$r_L^{(0)2} = 1 - \frac{2K_1 \ln(1+B)}{C} t \tag{A-13}$$

where

$$C = 1 - \frac{r_L^{(0)}}{r_f^{(0)}}$$

It is now convenient to define the reference time, t_r^*, to be the quasi-steady droplet lifetime. Then t = 1 when $r_L^{(0)} = 0$ which, for a given set of numbers B, ρ_L^*/ρ_f^* and C, defines K for the problem because

$$\frac{2K_1 \ln(1+B)}{C} = 1$$

Three convenient quantities are computed from Eq. (A-12):

$$T_r^{(0)} = \hat{m}_L^{(0)} (T^{(0)} + A) \tag{A-14a}$$

$$T_{rr}^{(0)} = \frac{\hat{m}^{(0)}(T^{(0)} + A)(\hat{m}^{(0)} - 2r)}{r^4} \tag{A-14b}$$

$$T_t^{(0)} = \frac{\hat{m}^{(0)}(T^{(0)} + A)}{2 r r_L^{(0)2}} \tag{A-14c}$$

Then from Eq's, (A-10c), (A-11b) and (A-14c)

$$\hat{m}^{(1)} = \hat{m}_L^{(1)} + \frac{\hat{m}^{(0)}}{2\, r_L^{(0)2}} \int_{r_L^{(0)}}^{r} \frac{(T^{(0)}+A)}{T^{(0)2}}\, r' dr' \qquad \text{(A-15)}$$

Using Eq's. (A-15), (A-11d) and (A-14a), Eq. (A-11a) may be integrated to yield

$$T^{(1)} = \frac{\hat{m}^{(1)}}{B_\infty}\left[\frac{1}{r_L^{(0)}} - \frac{e^{\hat{m}^{(0)}\left(\frac{1}{r_L^{(0)}} - \frac{1}{r}\right)}}{r}\right] - \frac{\hat{m}^{(0)} r_L^{(1)}}{r_L^{(0)2} B_\infty}$$

$$+ \frac{\hat{m}^{(0)} e^{-\hat{m}^{(0)}/r}}{2\, r_L^{(0)2}} \int_{r_L^{(0)}}^{r} \frac{T^{(0)} e^{\hat{m}^{(0)}/r'}}{r'^2} dr' \int_{r_L^{(0)}}^{r'} \frac{(T^{(0)}+A)}{T^{(0)2}}\, r'' dr'' \qquad \text{(A-16)}$$

Succeeding order solutions may be readily carried out since only the form of the inhomogeneous parts of Eq's. (A-11a) and(A-11b) will change. However, only the first order solution will be carried out here. The essential singularity at $r=0$ is excluded except for the final instant of vaporization. However, the singularity at $r_L^{(0)}=0$ will invalidate the expansion procedure unless the droplet vaporizes faster than at the quasi-steady rate. That is, unless $r_L(t) < r_L^{(0)}(t)$ the solution for $T^{(1)}$ blows up before the droplet actually disappears. Applying Eq. (A-11e)

$$\hat{m}_L^{(1)} = \frac{\dfrac{\hat{m}^{(0)} r_L^{(1)}}{r_L^{(0)}} - \dfrac{\hat{m}^{(0)} r_L^{(0)} (1+B) r_f^{(1)}}{r_f^{(0)2}} - \dfrac{\hat{m}^{(0)} B_\infty e^{-\hat{m}^{(0)}/r_f^{(0)}}}{2\, r_L^{(0)}} F_{r_L^{(0)}}^{r_f^{(0)}}}{1 + (1+B)\dfrac{r_L^{(0)}}{r_f^{(0)}}} \qquad \text{(A-17)}$$

where

$$F_{r_L^{(0)}}^{r_f^{(0)}} = \int_{r_L^{(0)}}^{r_f^{(0)}} \frac{T^{(0)} e^{\hat{m}^{(0)}/r'}}{r'^2} dr' \int_{r_L^{(0)}}^{r'} \frac{(T^{(0)}+A)}{T^{(0)2}} r'' dr''$$

It is now desirable to make the assumption that

$$\frac{r_L(t)}{r_f(t)} = \text{constant}$$

This is in accord with some observations on burning droplets or requires an approximately constant Reynolds number in the case of convective vaporization. This assumption leads to the result that

$$\frac{r_L^{(0)}}{r_f^{(0)}} = \frac{r_L^{(1)}}{r_f^{(1)}} = \cdots\cdot = 1-C \tag{A-18}$$

Finally, using Eq's. (A-11g), (A-13), (A-17), and (A-18) along with the quasi-steady solution

$$\hat{m}^{(0)} = \frac{r_L^{(0)} \ln(1+B)}{C}$$

and the boundary condition

$$r_L^{(1)}(0) = 0$$

the following solution for $r_L^{(1)}$ is obtained:

$$r_L^{(1)} = \frac{1}{4} \frac{B_\infty e^{-\frac{r_L^{(0)}}{r_f^{(0)}} \frac{\ln(1+B)}{C}}}{\left[1-(1+B)\frac{r_L^{(0)}}{r_f^{(0)}}\right](1-t)^{1/2}} \int_0^t \frac{F_{r_L^{(0)}(t')}^{r_f^{(0)}(t')}}{(1-t')^{1/2}} dt' \tag{A-19}$$

The condition for the expansion to remain regular is then

$$(1+B)\frac{r_L^{(0)}}{r_f^{(0)}} > 1$$

since the integrand is always positive.

Since the above analysis gives no information concerning the relaxation of the initial condition, this problem must be treated separately. Consideration will be given to a porous sphere continually wetted with liquid at the wet bulb temperature of a surrounding high temperature gas into which the sphere is suddenly thrust. Thus, the initial conditions are presumed known and the droplet radius remains constant in time $(r_L = 1)$. It is convenient to adopt

$$y = \frac{T^* - T_L^*}{T_f^* - T_L^*}$$

as the independent variable. Then the energy equation, the boundary conditions, and the initial condition become the following

$$y_\tau = -v\,y_r + \frac{1}{\rho r^2}\left(r^2 y_r\right)_r \tag{A-20}$$

$$y(1,t) = 0 \qquad y(r_f,t) = 1 \tag{A-21}$$

$$y(r,0) = a(r) \tag{A-22}$$

where t has been contracted by

$$\tau = \kappa t \tag{A-23}$$

It is desirable to convert Eq. (A-20) into a linear form so that the powerful methods of linear mathematics may be used. The assumption will be made that the density is not a function of time, but remains the function of distance as specified by the initial condition. This will clearly abandon the state equation, but through Eq. (A-4a) this requires that $v(r)$ is not a function of time. It is still hoped that the behavior of the relaxation of the initial condition to the steady state condition is not seriously affected by this simplification, at least for initial conditions which do not radically depart from the steady-state conditions.

Under this idealization it is possible to adapt a method developed by Frisch (13) to obtain a useful result. Eq. (A-20) may be written as

$$y_\tau = \mathcal{L}_r \left[y(r, t) \right] \tag{A-24}$$

where $\mathcal{L}_r$ is a linear operator in r defined by

$$\mathcal{L}_r = \frac{1}{\rho r^2 e^{m/r}} \frac{\partial}{\partial r} \left[r^2 e^{m/r} \frac{\partial}{\partial r} \right] \tag{A-25}$$

Define a steady-state solution given by

$$\mathcal{L}_r \left[y^{(s)}(r) \right] = 0 \tag{A-26}$$

under the boundary condition

$$y^{(s)}(1) = 0 \qquad y^{(s)}(r_f) = 1 \tag{A-27}$$

Define a relaxation time lag as

$$T(r) = -\frac{\int_0^\infty \mathcal{Q}_r[y - y^{(s)}]\,dt}{\mathcal{Q}_r[y^{(s)}]} \tag{A-28}$$

where $\mathcal{Q}_r$ is some other linear operator. If $\mathcal{Q}_r \equiv 1$, $T(r)$ describes a typical relaxation time that the temperature takes to come to steady-state. The interest here primarily concerns heat transfer so that from here on $\mathcal{Q}_r = \frac{\partial}{\partial r}$ will be considered.

Let $u(r,t) = y(r,t) - y^{(s)}(r)$ so that

$$y_\tau = \mathcal{L}_r[u(r,t)] \tag{A-29}$$

is subject to

$$u(1,t) = u(r_f,t) = 0 \tag{A-30}$$

Defining a Green's function

$$\mathcal{L}_r[G(r,r')] = -\delta(r-r') \tag{A-31}$$

subject to

$$G(1,r') = G(r_f,r') = 0 \tag{A-32}$$

yields as a solution to Eq's. (A-29) and (A-30)

$$u(r,t) = -\int_1^{r_f} G(r,r')\,y_\tau(r',\tau)\,dr' \tag{A-33}$$

Substituting Eq. (A-33) into Eq. (A-28), using Eq's. (A-22) and (A-26), and providing the order of integration may be justifiably changed,

$$T(r) = \frac{\int_1^{r_f} \frac{\partial}{\partial r}\left[G(r,r')\right]\left[y^{(s)}(r') - a(r')\right] dr'}{dy^{(s)}/dr} \tag{A-34}$$

Thus the problem is reduced to finding a solution to Eq's. (A-31) and (A-32). The Green's function may be written

$$G(r,r') = \frac{\rho' r'^2 e^{\hat{m}/r'}}{\hat{m}(e^{-\hat{m}} - e^{-\hat{m}/r_f})}\Big[(e^{-\hat{m}/r} - e^{-\hat{m}})(e^{-\hat{m}/r'} - e^{-\hat{m}/r_f})H(r'-r) + (e^{-\hat{m}/r'} - e^{-\hat{m}})(e^{-\hat{m}/r} - e^{-\hat{m}/r_f})H(r-r')\Big] \tag{A-35}$$

where H is the Heaviside unit operator. The greatest interest is at $r=1$ so that substituting Eq. (A-35) into Eq. (A-34) and evaluating at ,

$$T(1) = \frac{1}{\left.\frac{dy^{(s)}}{dr}\right)_{r=1}\left[1 - e^{\hat{m}(1-\frac{1}{r_f})}\right]}\int_1^{r_f} \rho' r'^2\left[1 - e^{\hat{m}(\frac{1}{r'}-1)}\right]\left[y^{(s)}(r') - a(r')\right]dr' \tag{A-36}$$

The steady-state solution may be obtained from the previous problem:

$$y^{(s)} = \frac{1}{B}\left[e^{\hat{m}(1-\frac{1}{r})} - 1\right]$$

so that the time lag is

$$T(1) = \frac{\left(\frac{1}{r_f} - 1\right)B}{\ln(1+B)}\int_1^{r_f} \rho' r'^2\left[\frac{y^{(s)}(r')}{B\,y^{(s)}(r') + 1}\right]\left[y^{(s)}(r') - a(r')\right]dr' \tag{A-37}$$

This integral blows up if r_f is extended to large values of r. However, practically r_f is kept finite by convection, burning, or space limitations. Converting this time lag to a physical basis by Eq. (A-23),

$$\frac{t^*_{Ic}}{t^*_d} = \frac{T(1)}{K}$$

APPENDIX B: THE VAPORIZATION RATE RESPONSE OF A PLANE LIQUID SURFACE TO PRESSURE AND TEMPERATURE OSCILLATIONS

Consider a one-dimensional vaporization model with the liquid surface always maintained at x = 0. Assume a boundary $x = x_f$ may be imposed where the temperature and mass fraction of the liquid vapor are known. This may correspond to a flame surface, the outer edge of a boundary layer, or some type of mass absorbing, heat releasing surface. It will further be assumed that this boundary remains at a constant position in time. The reference quantities on p. 8 of Chapter I are

$$\tau_r^* = \frac{\rho_f^* c_p^* x_f^{*2}}{\bar{\lambda}^*} \qquad U_r^* = \frac{\bar{\lambda}^*}{\rho_f^* c_p^* x_f^*}$$

$$x_r^* = x_f^* \qquad \rho_r^* = \rho_f^*$$

$$T_r^* = T_f^*$$

Retain assumptions 3 , 4 , 9 , and 10 of pp. 12 and 13 of Chapter I. Then allowing the average values of the transport properties to be time varying as explained on p. 34 the equations for the diffusion layer are

<u>Continuity</u> $$\rho_t + m_x = 0 \tag{B-1}$$

<u>Mass Fraction</u> $$\rho Y_{K_t} + m Y_{K_x} = \frac{1}{Le} Y_{K_{xx}} \tag{B-2}$$

Energy $\rho T_t + m T_x = K T_{xx} + \frac{\gamma - 1}{\gamma} P_t$ (B-3)

Diffusion $\frac{1}{Le} Y_{K_x} = m Y_K - m_K$ (B-4)

State $P = \rho T$ (B-5)

In the steady-state $Le = K = 1$. The heat transfer in the liquid must be considered, so a semi-infinite liquid surface is treated. The energy equation for the liquid is

$$t_4 T_{L_t} = T_{L_{xx}} \tag{B-6}$$

The boundary conditions are

$$
\begin{aligned}
&T(0,t) = T_o(t) \\
&T(1,t) = T_f(t) \\
&T_L(0,t) = T_o(t) \\
&K T_x(0,t) = \frac{m(0,t)}{B_\infty} + \lambda_{L/g} T_{L_x}(0,t) \\
&\frac{1}{Le} Y_{F_x}(0,t) = m(0,t) \left[Y_{F_L}(t) - 1 \right] \\
&Y_{F_L}(t) = f\left[T_o(t) \right] \quad \text{--- equilibrium}
\end{aligned}
\tag{B-7}
$$

It is assumed for simplicity that B_∞ is constant in time, i.e., the latent heat is not varying.[1] Although it must vary if $T_o(t)$ is non-steady it will only yield the same

---

1. B_∞ is based to T_f^*.

order of variation in the results as the variation of $T_o(t)$ It is the partial purpose of this section to show that T_o may be considered stationary in time with only a small error. It is assumed that the liquid is at its "wet bulb" temperature in the steady state so that the steady problem has all heat transfer going toward vaporization, not liquid heat-up.

Now the existence of a periodic solution to a perturbed problem is investigated by assuming

$$
\begin{aligned}
T &= \overline{T}(x) + \sigma(x)\, e^{i\omega t} \\
Y_K &= \overline{Y}_K(x) + y_K(x)\, e^{i\omega t} \\
p &= 1 + \varphi\, e^{i\omega t} \\
\rho &= \overline{\rho}(x) + \rho(x)\, e^{i\omega t} \\
K &= 1 + \chi\, e^{i\omega t} \\
m &= \overline{m} + M(x)\, e^{i\omega t} \\
Le^{-1} &= 1 + \mathcal{L}\, e^{i\omega t}
\end{aligned}
\qquad (B-8)
$$

where the time dependent perturbation quantities are considered to be arbitrarily small so that products and squares of perturbation quantities may be neglected. First, by the use of Eq's. (B-5) and (B-1) Eq. (B-3) may be directly integrated with respect to the space variable to yield

$$\kappa T_x - mT = \frac{x}{\gamma}\rho_t + m(0,t)\left[\frac{1}{B_\infty} - T_0(t)\right] + \lambda_{L/g}\, T_{L_x}(0,t) \tag{B-9}$$

By use of Eq. (B-1), m may be eliminated from Eq. (B-9)

$$\kappa\left[TT_{xx} - T_x^2\right] - \rho T_t = \frac{\rho_t}{\gamma}\left[(1-\gamma)T - xT_x\right] - m(0,t)\,T_x\left[\frac{1}{B_\infty} - T_0(t)\right] - \lambda_{L/g}\, T_x T_{L_x}(0,t) \tag{B-10}$$

Introducing Eq's. (B-8) into Eq's. (B-1), (B-2), B-5), (B-6), and (B-10) the steady state set is obtained

$$\bar{m}' = 0 \tag{B-11}$$

$$\bar{m}\,\bar{Y}_K' = \bar{Y}_K'' \tag{B-12}$$

$$\bar{\rho}\,\bar{T} = 1 \tag{B-13}$$

$$\bar{T}\,\bar{T}'' - (\bar{T}')^2 = -\bar{m} A \bar{T}' \tag{B-14}$$

$$\bar{T}_L'' = 0 \tag{B-15}$$

and the perturbation set

$$i\omega\,\xi + M_x = 0 \tag{B-16}$$

$$y_K'' - \bar{m}\, y_K' - \frac{i\omega}{\bar{T}} y_K = M\, \bar{Y}_K' - \alpha\, \bar{Y}_K'' \tag{B-17}$$

$$\xi \bar{T} + \bar{\xi}\, \sigma = \varphi \tag{B-18}$$

$$\sigma'' - \left[2\bar{m} + \frac{\bar{m}A}{\bar{T}}\right]\sigma' + \left[\frac{\bar{m}^2(\bar{T}+A)}{\bar{T}} - \frac{i\omega}{\bar{T}}\right] =$$
$$\bar{m}^2 \frac{(\bar{T}+A)}{\bar{T}}\left[\sigma_L(0) - \lambda_{L/g}\frac{\bar{T}}{\bar{T}}\sigma_L'(0) + A\left(\kappa - \frac{M_F(0)}{\bar{m}}\right)\right]$$
$$+ \frac{i\omega\varphi}{\gamma}\left[1 - \gamma - x\bar{m}\left(1 + \frac{A}{\bar{T}}\right)\right] \tag{B-19}$$

$$i\omega t_4\, \sigma_L - \sigma_L'' = 0 \tag{B-20}$$

The solution to the liquid phase problem, Eq's. (B-15) and (B-20) is easily written

$$\bar{T}_L = \tau$$
$$\sigma_L = \sigma_L(0)\, e^{\sqrt{i\omega t_4}\; x} \tag{B-21}$$

so that if $\sigma_L(0) = 0$ there is no heat transfer to the liquid even in the time dependent case. From the equilibrium interface condition in Eq's. (B-7) the perturbation form may be written as

$$y_F(0) = b\, \sigma_L(0)$$

For normal fuels the slope of the partial pressure vs. liquid temperature at the saturated vapor line is large

(b is of order 10). Therefore since only reasonable magnitudes of $\mathcal{Y}_F(o)$ are expected to be necessary to carry out the perturbed mass flow, $\sigma_L(o)$ should be small. Assuming $\sigma_L(o)=0$ the mass fraction problem is decoupled from the temperature problem if the only interest is in the mass flow perturbation at the surface. This problem will be investigated first. In Eq's. (B-7) assume

$$T_{\varsigma}(t) = 1 + \frac{P-1}{P} e^{i\omega t} \varphi \tag{B-22}$$

or a polytropic temperature perturbation at the outer edge of the film. Further assume that the average gas thermal conductivity is proportional to the $\hat{\omega}$ power of the outer temperature so that

$$k = \hat{\omega}\left(\frac{P-1}{P}\right) \tag{B-23}$$

The solution to the steady state gas phase problem, Eq's. (B-11), (B-12), and (B-14), is

$$\bar{m} = \ln(1+B)$$

$$\bar{T} + A = \frac{e^{\bar{m}x}}{B_\infty}$$

$$\bar{Y}_F - 1 = (\bar{Y}_{F_L} - 1)e^{\bar{m}x} \tag{B-24}$$

Under the present approximation Eq. (B-19) is rewritten

$$\sigma'' - \left[2\bar{m} + \frac{\bar{m}A}{\bar{T}}\right]\sigma' + \left[\frac{\bar{m}^2(\bar{T}+A)}{\bar{T}} - \frac{i\omega}{\bar{T}}\right] =$$

$$\frac{\bar{m}^2 A(\bar{T}+A)}{\bar{T}}\left(s - \frac{M_F(0)}{\bar{m}}\right) + \frac{i\omega\varphi}{\gamma}\left[1-\gamma - x\bar{m}\left(1+\frac{A}{\bar{T}}\right)\right] \qquad \text{(B-25)}$$

By defining new dependent and independent variables

$$\eta = \frac{\sigma}{\bar{T}+A}$$

$$\xi = \frac{A}{\bar{T}+A} \qquad \text{(B-26)}$$

Eq. (B-25) is transformed to

$$\xi(1-\xi)\eta'' + \eta' - \frac{i\omega}{\bar{m}^2 A}\eta = s - \frac{M_F(0)}{\bar{m}}$$

$$+ \frac{i\omega}{\bar{m}^2 A}\frac{\varphi}{\gamma}\left[(1-\gamma)(1-\xi) + \ln\left(\frac{\xi}{AB_\infty}\right)\right] \qquad \text{(B-27)}$$

The homogeneous part of Eq. (B-27) is of the hypergeometric type and has the two linearly independent solutions

$$\eta_1 = {}_2F_1[\alpha, \beta; \gamma'; \xi] \qquad \alpha = -\frac{1}{2} - \sqrt{\frac{1}{4} - in}$$

$$\beta = -\frac{1}{2} + \sqrt{\frac{1}{4} - in}$$

$$\gamma' = 1$$

$$\eta_2 = \eta_1 \ln \xi + \sum_{r=1}^{\infty} c_r \xi^r \qquad n = \omega/\bar{m}^2 A$$

(B-28)

The particular form of Eq. (B-27) and transformation (B-26) was chosen since $|A| < 1$ for most rocket propellants and the series development, Eq. (B-28), converges in the entire ξ interval. The recurrence relations for Eq's (B-28) are

$$\eta_1 = \sum_{r=0}^{\infty} S_r \xi^r$$

$$S_r = S_{r-1}\left[\frac{in + (r-1)(r-2)}{r^2}\right]$$

$$C_{r+1} = \frac{r(r-1) + in}{(r+1)^2} C_r + \frac{3r - 1 - 2in}{(r+1)^3} S_r$$

$$S_0 = 1$$

(B-29)

A particular solution to Eq. (B-27) is obtained by inspection

$$\eta_p = \frac{i}{n}\left(\kappa - \frac{M_F(0)}{\overline{m}}\right) + \frac{\varphi}{\gamma}\left\{(\gamma-1)\left(1 - \xi + \frac{i}{n}\right) + \frac{i}{n} - \ln\left(\frac{\xi}{AB_\infty}\right)\right\} \qquad \text{(B-30)}$$

The error made in actually computing the series, Eq's. (B-29), may be bounded by standard methods but these details will not be presented here. The complete solution to the temperature problem may be written

$$\eta = \mathcal{A}\,\eta_1 + \mathcal{B}\,\eta_2 + \eta_p \qquad \text{(B-31)}$$

where $\mathcal{A}$ and $\mathcal{B}$ are integration constants. However, η_P contains the unknown $M_F(0)$. Then the heat transfer condition at the liquid in Eq. (B-7) is employed so that

$$\frac{M_F(0)}{\bar{m}} = K + \frac{B_\infty \sigma'(0)}{\bar{m}} \tag{B-32}$$

Upon application of the two temperature boundary conditions $\sigma(0) = 0$ and $\sigma(1) = \frac{P-1}{P}\psi$ Eq's. (B-31) and (B-32) form a system of three (complex) equations in the three (complex) unknowns $\mathcal{A}$, $\mathcal{B}$, and $M_F(0)$. The above procedure fails when $A \equiv 0$ in which case the appropriate transformations are

$$\sigma = T\eta$$
$$\xi = 1 - \frac{\tau}{T}$$

and a solution may be developed similar to above.

With the aid of this simple model it is now desirable to relax the assumption that $\sigma_L(0) = 0$; if this relaxation makes only a small difference in the results the effect may be neglected in more complex models of the burning process. Once this assumption is relaxed the mass fraction Eq. (B-17) must also be considered which introduces a great deal more complexity in computation. Again a series development under appropriate transformations may be obtained for the homogeneous part of Eq. (B-17). The particular solution is obtained as

$$\mathcal{Y}_{F_P} = \int_0^x \bar{m}^2 (\bar{Y}_F - 1)\left(\frac{M}{\bar{m}} - \mathcal{L}\right)\left[\mathcal{Y}_{F_1}(x')\mathcal{Y}_{F_2}(x) - \mathcal{Y}_{F_1}(x)\mathcal{Y}_{F_2}(x')\right] \frac{dx'}{W(x')}$$

where $W(x')$ is the Wronskian

$$W(x') = \mathcal{Y}_{F_1}(x')\mathcal{Y}_{F_2}'(x') - \mathcal{Y}_{F_2}(x')\mathcal{Y}_{F_1}'(x')$$

and

$$\mathcal{L} = A$$

is assumed.

From Eq. (B-16)

$$M = -i\omega \int_0^x \frac{\varphi}{T}\left(1 - \frac{\sigma}{\rho T}\right) dx' \quad + M_F(0)$$

and it is clear that the $\mathcal{Y}_F$ equation is coupled with Eq. (B-19). The solution to Eq. (B-17) may be written

$$\mathcal{Y}_F = \mathcal{E}\mathcal{Y}_{F_1} + \mathcal{F}\mathcal{Y}_{F_2} + \mathcal{Y}_{F_P} \tag{B-33}$$

where $\mathcal{E}$ and $\mathcal{F}$ are undetermined constants. The particular solution, Eq. (B-30), now contains terms in $\sigma_L(0)$ and $\sigma_L'(0)$ from Eq. (B-19). $\sigma_L'(0)$ is related to $\sigma_L(0)$ by Eq. (B-21). $\mathcal{Y}_F(0)$ is related to $\sigma_L(0)$ by the equilibrium condition. The surface mass diffusion relation in Eq. (B-7) is

$$\mathcal{Y}_F'(0) + \mathcal{L}\,\bar{Y}_F'(0) - \bar{m}\,\mathcal{Y}_F(0) + M_F(0)\left(1 - \bar{Y}_{F_L}\right) = 0 \tag{B-34}$$

Now Eqs. (B-31) and (B-34) form a system of six (complex) equations in the six (complex) unknowns $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$, $\mathcal{F}$, $M_F(0)$ and $\sigma_L(0)$; six equations arise becuase Eq's. (B-31) and (B-33) are both evaluated at two endpoints under the endpoint boundary conditions. During the computation only the case $A=0$ has been looked at since this parameter is usually quite small for high temperature vaporization.

One further assumption has been checked with this model. It has been assumed above that the liquid-gas interface is in equilibrium; therefore the kinetics of evaporation have been ignored. If the evaporation process cannot take place as fast as demanded by this solution, these results are in considerable error. The evaporation rate, is governed primarily by the difference between the actual partial pressure of the fuel vapor above the surface of the liquid and the equilibrium partial pressure corresponding to the liquid surface temperature. Other parameters than those included in the main theory must enter since a molecular collision process is governing. This may be avoided by merely assuming that in the steady state there is a fixed percentage mass fraction drop between the equilibrium surface and actual gas. In the unsteady state a new unknown is introduced, the actual gas fuel mass fraction and a new equation

$$\frac{M_F(0)}{\overline{m}} = \mathcal{Y}_{F_L}(0)\ \text{equilibrium} - \mathcal{Y}_F(0)\ \text{actual}$$

as the kinetic equation. The effect of this has been carried out for a severe, but probable, case of 10% mass fraction drop in the steady state.

APPENDIX C: THE BURNING RATE RESPONSE OF FUEL DROPLETS IN AN OXIDIZING ATMOSPHERE DURING SPHERICALLY SYMMETRIC PRESSURE AND TEMPERATURE OSCILLATIONS

Consider the configuration of Figure 1a under assumptions 1, and 3-10 on pp. 12 and 13 of Chapter I. Choosing the reference quantities

$$x_r^* = r_{L_0}^* \qquad t_r^* = \frac{\rho_f^* c_p^* r_{L_0}^{*2}}{\bar{\lambda}^*}$$

$$\rho_r^* = \rho_f^* \qquad T_r^* = T_f^*$$

$$u_r^* = \frac{\bar{\lambda}^*}{\rho_f^* c_p^* r_{L_0}^*}$$

the dimensionless equations for the gas field become

$$\hat{m}_r + r^2 \rho_t = 0 \tag{C-1}$$

$$\rho r^2 Y_{K_t} + \hat{m} Y_{K_r} = (r^2 Y_{K_r})_r \tag{C-2}$$

$$\rho r^2 T_t + \hat{m} T_r = (r^2 T_r)_r + \frac{\gamma - 1}{\gamma} r^2 p_t \tag{C-3}$$

$$p = \rho T \tag{C-4}$$

$$r^2 Y_{K_r} = \hat{m} Y_K - \hat{m}_K \tag{C-5}$$

Now investigate the existence of a periodic solution to the perturbed problem by introducing

$$
\begin{aligned}
T &= \bar{T}(r) + \sigma(r)e^{i\omega t} \\
P &= 1 + \varphi e^{i\omega t} \\
\rho &= \bar{\rho}(r) + \xi(r)e^{i\omega t} \\
Y_K &= \bar{Y}_K(r) + y_K(r)e^{i\omega t} \\
\hat{m} &= \hat{\bar{m}} + \hat{M}(r)e^{i\omega t} \\
r_f &= \bar{r}_f + R_f e^{i\omega t}
\end{aligned}
\tag{C-6}
$$

where the time dependent perturbation quantities are arbitrarily small. Note that no perturbation in the transport properties is being considered. Substituting Eq. (C-6) into Eq's. (C-1) - (C-5) and neglecting products and squares of perturbation quantities yields the steady state set

$$\hat{\bar{m}}' = 0 \tag{C-7}$$

$$\hat{\bar{m}}\,\bar{T}' = (r^2\bar{T}')' \tag{C-8}$$

$$\hat{\bar{m}}\,\bar{Y}_K' = (r^2\bar{Y}_K')' \tag{C-9}$$

$$\bar{\rho}\,\bar{T} = 1 \tag{C-10}$$

$$r^2\bar{Y}_K' = \hat{\bar{m}}\,\bar{Y}_K - \hat{\bar{m}}_K \tag{C-11}$$

and the perturbation set

$$\hat{M}' + i\omega r^2 \xi = 0 \tag{C-12}$$

$$\sigma'' + \left(\frac{2}{r} - \frac{\hat{m}}{r^2}\right)\sigma' - \frac{i\omega\sigma}{\bar{T}} = \frac{\hat{M}}{r^2}\bar{T}' - i\omega\,\frac{(\gamma-1)}{\gamma}\,\varphi \tag{C-13}$$

$$y_K'' + \left(\frac{2}{r} - \frac{\hat{m}}{r^2}\right)y_K' - \frac{i\omega y_K}{\bar{T}} = \frac{\hat{M}\,\bar{Y}_K'}{r^2} \tag{C-14}$$

$$\xi\bar{T} + \bar{\rho}\,\sigma = \varphi \tag{C-15}$$

$$r^2 y_K' = \hat{m}\, y_K + \hat{M}\bar{Y}_K - \hat{M}_K \tag{C-16}$$

The boundary conditions for the collapsed flame zone model with an equilibrium liquid-gas interface at the droplet surface are

$$r_L^2 Y_{F_r}(r_L) = \left[\hat{m}\,Y_{F_L} - \hat{m}\right]_{r_L} \qquad r_L^2 T_r(r_L) = \frac{\hat{m}(r_L)}{B_f} + \text{heat to droplet}$$

$$Y_F(r_L) = Y_{F_L}(t) \qquad T(r_L) = \tau(t)$$

$$T_r(r_f-) - T_r(r_f+) - \frac{\hat{m}_F(r_f)}{r_f^2}\, q = 0$$

$$j\,Y_{F_r}(r_f) + Y_{O_r}(r_f) = 0 \tag{C-17}$$

$T(r_f) = T_f$ is continuous

$Y_F(r_f) = Y_O(r_f) = 0$

$T(\infty) = T_\infty$ specified

$Y_O(\infty) = Y_{O_\infty}$ specified

It is fortuitous that for many hydrocarbons

$$A = \frac{1}{B_f} - \tau \ll 1$$

since assuming $A = 0$ simplifies the algebraic manipulation. Then one parameter is lost by replacing B_f with $1/\tau$. The solution to the steady state problem yields the following familiar equations:

Fuel Side

$$\bar{T} = \bar{\tau}\, e^{\hat{\bar{m}}\left(\frac{1}{\bar{r}_L} - \frac{1}{\bar{r}}\right)}$$

$$\bar{Y}_F - 1 = (\bar{Y}_{F_L} - 1)\, e^{\hat{\bar{m}}\left(\frac{1}{\bar{r}_L} - \frac{1}{\bar{r}}\right)}$$

Oxidizer Side

$$\bar{T} = e^{\hat{\bar{m}}\left(\frac{1}{\bar{r}_f} - \frac{1}{\bar{r}}\right)}(1 - q) + q$$

$$\bar{Y}_o = -j\left[1 - e^{\hat{\bar{m}}\left(\frac{1}{\bar{r}_f} - \frac{1}{\bar{r}}\right)}\right]$$

General

$$1 + \frac{\bar{Y}_{o\infty}}{j} = \frac{\bar{T}_\infty - q}{1 - q}$$

$$\frac{\bar{r}_f}{\bar{r}_L} = 1 + \frac{\ln(1/\bar{\tau})}{\ln\left(1 + \frac{\bar{Y}_{o\infty}}{j}\right)}$$

$$\hat{\bar{m}} = \bar{r}_L \ln\left[\frac{1}{\bar{\tau}} + 1 + \frac{Y_{o\infty}}{j}\right] \tag{C-18}$$

These should be compared with Eq's. (1.25) and (1.26) bearing in mind the difference in reference quantities. The boundary conditions for the unsteady problem deserve some attention since the flame (line heat source) is moving in time. It is assumed that any quantity may be evaluated at the flame by continuation from the steady state region. Therefore, if $\mathcal{F}(r)$ is the space-dependent perturbation

$$f(r_f(t),t) = \bar{f}(\bar{r}_f) + \left.\frac{d\bar{f}}{dr}\right)_{\bar{r}_f} R_f\, e^{i\omega t} + \mathcal{J}(\bar{r}_f)\, e^{i\omega t}$$

The derivative is taken on the side of the flame of interest even though this derivative may be discontinuous at the flame. This is a common procedure in perturbation problems and its validity will not be discussed further. Then the conditions (C-23) for the perturbation problem become

$$y_F(\bar{r}_L) = y_{F_L} \qquad \sigma(\bar{r}_L) = 0$$

$$\bar{r}_L^{\,2}\, y_F'(\bar{r}_L) = \hat{\bar{m}}\, y_{F_L} + \hat{M}_L(\bar{Y}_{F_L} - 1) \qquad \sigma'(r_L) = \frac{\tau \hat{M}_L}{\bar{r}_L^2}$$

$$\sigma'(r_f-) - \sigma'(r_f+) + [\bar{T}''(r_f-) - \bar{T}''(r_f+)]R_f - M_F(\bar{r}_f)\, q = 0$$

$$j\, y_F'(\bar{r}_f) + y_o'(\bar{r}_f) + [j\, \bar{Y}_F''(\bar{r}_f) + \bar{Y}_o''(\bar{r}_f)]\, R_f = 0$$

$$y_F(\bar{r}_f) + R_f\, \bar{Y}_F'(\bar{r}_f) = y_o(\bar{r}_f) + R_f\, \bar{Y}_o'(\bar{r}_f) = 0$$

$$\sigma(\bar{r}_f-) + R_f\, \bar{T}'(\bar{r}_f-) = \sigma(\bar{r}_f+) + R_f\, \bar{T}'(r_f+)$$

$$\sigma(\infty) = \frac{\gamma-1}{\gamma}\, \bar{T}_\infty\, \varphi$$

$$y_o(\infty) = 0 \tag{C-19}$$

It is assumed above that all heat transfer at the droplet surface goes toward supplying the latent heat of vaporization and that none changes the droplet temperature. In the steady state this is equivalent to assuming the attainment of an equilibrium "wet bulb" temperature for the droplet. In the unsteady problem it is assumed that this contribution is small. The validity of this assumption is discussed in Appendix B. It is also assumed that the oscillation is isentropic at infinity. As will be seen this is the only consistent possibility.

One of the most interesting quantities is $M_F(\bar{r}_f)\bar{r}_f^2 + 2\bar{m} R_f/\bar{r}_f^2$; this is the perturbation in the mass burning rate. To obtain this function the complete droplet problem must be solved and not just the energy equation considered as in Appendix B. With this in mind an exact solution of Eq's. (C-12)-(C-16) subject to conditions (C-19) is attempted.

It is usual in diffusion problems that an expansion in powers of the frequency is a convergent one, although it is not practically useful for high frequencies. Therefore, such an expansion is attempted

$$\sigma = \sum_{n=0}^{\infty} (i\omega)^n \sigma^{(n)}$$

$$y_K = \sum_{n=0}^{\infty} (i\omega)^n y_K^{(n)}$$

$$\hat{M} = \sum_{n=0}^{\infty} (i\omega)^n \hat{M}^{(n)} \qquad \text{(C-20)}$$

Substituting the expansion in Eq's. (C-12)-(C-16) and collecting like powers of the frequency

$$
\begin{aligned}
&M^{(0)\prime} = 0\\
&\sigma^{(0)\prime\prime} + \left(\frac{2}{r} - \frac{\hat{m}}{r^2}\right)\sigma^{(0)\prime} = \frac{\hat{M}^{(0)}\bar{T}'}{r^2}\\
&y_K^{(0)\prime\prime} + \left(\frac{2}{r} - \frac{\hat{m}}{r^2}\right)y_K^{(0)\prime} = \frac{\hat{M}^{(0)}\bar{Y}_K'}{r^2}
\end{aligned}
\qquad \text{(C-21)}
$$

$$
\begin{aligned}
&\hat{M}^{(1)\prime} + \frac{r^2}{\bar{T}}\left(\varphi - \frac{\sigma^{(0)}}{\bar{T}}\right) = 0\\
&\sigma^{(1)\prime\prime} + \left(\frac{2}{r} - \frac{\hat{m}}{r^2}\right)\sigma^{(1)\prime} = \frac{\hat{M}^{(1)}\bar{T}'}{r^2} - \left(\frac{\gamma-1}{\gamma}\right)\varphi + \frac{\sigma^{(0)}}{\bar{T}}\\
&y_K^{(1)\prime\prime} + \left(\frac{2}{r} - \frac{\hat{m}}{r^2}\right)y_K^{(1)\prime} = \frac{\hat{M}^{(1)}\bar{Y}_K'}{r^2} + \frac{y_K^{(0)}}{\bar{T}}
\end{aligned}
\qquad \text{(C-22)}
$$

and for $n > 1$

$$
\begin{aligned}
&\hat{M}^{(n)\prime} - \frac{r^2\sigma^{(n-1)}}{\bar{T}^2} = 0\\
&\sigma^{(n)\prime\prime} + \left(\frac{2}{r} - \frac{\hat{m}}{r^2}\right)\sigma^{(n)\prime} = \frac{\hat{M}^{(n)}\bar{T}'}{r^2} + \frac{\sigma^{(n-1)}}{\bar{T}}\\
&y_K^{(n)\prime\prime} + \left(\frac{2}{r} - \frac{\hat{m}}{r^2}\right)y_K^{(n)\prime} = \frac{\hat{M}^{(n)}\bar{T}'}{r^2} + \frac{y_K^{(n-1)}}{\bar{T}}
\end{aligned}
\qquad \text{(C-23)}
$$

It will, however, be shown that this procedure diverges. An integrating factor for the σ and y_K equations, (C-21)-(C-27), is

$$r^2 e^{\hat{m}/r}$$

Eq's(C-21) represent the quasi-steady perturbations which must have a solution since the steady state is stable to a static perturbation. In fact

$$\sigma^{(0)\prime} r^2 e^{\hat{\hat{m}}/r} + \text{constant} = \hat{M}_L \int \bar{T}' e^{\hat{\hat{m}}/r} dr$$

and since $\bar{T}' \propto 1/r^2$ this integral obviously converges if taken to $r = \infty$. Consider, however, the solution to the problem of first order in frequency. $\hat{M}^{(1)}$ blows up as r^3 as $r \longrightarrow \infty$. The solution for $\sigma^{(1)}$ may be formally written

$$\sigma^{(1)} + \text{constant} \int \frac{e^{-\hat{\hat{m}}/r}}{r^2} dr =$$

$$\int \frac{e^{-\hat{\hat{m}}/r}}{r^2} dr \int \hat{M}^{(1)} \bar{T}' e^{\hat{\hat{m}}/r'} dr'$$

$$+ \int \frac{e^{-\hat{\hat{m}}/r}}{r^2} dr \int \left[\frac{\sigma^{(0)}}{\bar{T}} - \left(\frac{\gamma - 1}{\gamma} \right) \varphi \right] r'^2 e^{\hat{\hat{m}}/r'} dr'$$

However, the integrals on the right hand side blow up as r to the first power and also do not cancel one another. All subsequent powers of frequency yield solutions which blow up as r^n where n is the power of frequency considered. The conclusion is reached that there is no solution for arbitrary frequency if the outer boundary is cast to infinity. The significance of this is discussed in Section B of Chapter II.

There is one solution for this problem which can be obtained, the case in which the frequency becomes large. The frequency as used here is the ratio of a diffusion time to a cycle time, and the physical significance of a large frequency is the inability of the diffusional processes to keep pace with the oscillating field. Time rates of storage (mass, energy, etc.) become so large that the diffusion mechanisms cannot smooth out the field. As such, rapid changes in physical quantities take place only near the boundaries and these quantities tend to be very uniform away from these boundaries. If, then, the boundary condition at infinity is a natural condition for the field in the absence of diffusion it will tend to be the uniform condition established throughout the field in the case of high frequency. Such a condition is an isentropic oscillation "at infinity". It is now reasonable to expect that an asymptotic solution for high frequency will not blow up as finite power in r if properly treated.

For $\alpha = \sqrt{i\omega}$ assume

$$\hat{M} = \sum_{2}^{-\infty} \alpha^n \hat{M}^{(-n)}(r)$$

$$y_F = G(\alpha,r)\, e^{\alpha \int_{r_f}^{r} \frac{dr'}{\bar{T}^{1/2}}} + H(\alpha,r)\, e^{-\alpha \int_{r_i}^{r} \frac{dr'}{\bar{T}^{1/2}}} + K(\alpha,r)$$

$$y_o = L(\alpha,r)\, e^{-\alpha \int_{r_f}^{r} \frac{dr'}{\bar{T}^{1/2}}} + N(\alpha,r)$$

$$G = \sum_{0}^{-\infty} \alpha^n G^{(-n)}(r) \tag{C-24}$$

with similar assumed forms for σ and H, K, L, and N. For large α the first two terms for y_F and the first for y_O are important only near the liquid and flame boundaries. The terms die off exponentially and match no power of α in K or N. Therefore, K and N must be the solution in the majority of the field. The exponential solutions will be picked to satisfy the homogeneous differential equations and K and N will be particular solutions of the full differential equations. This is a common procedure of inner and outer expansions and is required since K and N developed in powers of $1/\alpha$ will not be able to satisfy all the boundary conditions. This is a singular perturbation problem with wild behavior taking place near the boundaries. The problem will not be done in complete generality, but only a sketch will be given to obtain the results for the leading power of frequency. Some of the actual analytical forms will not have to be developed. This procedure is completed in much more detail in Chapter IV for a more complex system. Formally expanding K and N in powers of $1/\alpha$

$$K^{(0)} = (1-\bar{Y}_F)\bar{T}\hat{\bar{m}}\frac{\hat{M}^{(-2)}}{r^4}$$

$$K^{(1)} = (1-\bar{Y}_F)\bar{T}\hat{\bar{m}}\frac{\hat{M}^{(-1)}}{r^4}$$

$$K^{(2)} = (1-\bar{Y}_F)\bar{T}\hat{\bar{m}}\frac{\hat{M}^{(0)}}{r^4} + \bar{T}\left\{\left[(1-\bar{Y}_F)\bar{T}\hat{\bar{m}}\frac{\hat{M}^{(-2)}}{r^4}\right]'' + \left(\frac{2}{r} - \frac{\hat{\bar{m}}}{r^2}\right)\left[(1-\bar{Y}_F)\bar{T}\hat{\bar{m}}\frac{\hat{M}^{(-2)}}{r^4}\right]'\right\}$$

$$N^{(0)} = -(\bar{Y}_O + j)\bar{T}\hat{\bar{m}}\frac{\hat{M}^{(-2)}}{r^4} \tag{C-25}$$

and so on recursively,

Now

$$\hat{M} = \hat{M}_L - \alpha^2 \int_{r_L}^{r} \frac{r'^2}{T} \left(\varphi - \frac{\sigma}{T} \right) dr' \qquad \text{(C-26)}$$

and since σ has a similar expansion to y_K, it may be shown that

$$\hat{M}^{(-2)} = - \int_{r_L}^{r} \frac{r'^2}{T} \frac{\varphi}{\gamma} \left[1 - \frac{(r_L^3 - r^3)}{r^4} \hat{\hat{m}} \right] dr' \qquad \text{(C-27)}$$

assuming $\hat{M}_L$ is $O[\alpha]$ or less. $\hat{M}^{(-2)}$ blows up as r^3 as $r \to \infty$. This behavior is sufficiently suppressed so $N(\alpha, r) \to 0$ as $r \to \infty$ for all orders in α. Thus, the outer boundary condition on y_o is satisfied. Construction of the full solution near the surface and flame boundaries is now attempted.

Consider first the droplet surface. Expanding y_{F_L},

$$H^{(n)}(r_L) = y_{F_L}^{(n)} - K^{(n)}(r_L)$$

The mass transfer condition yields

$$H'(r_L) - \alpha \frac{H(r_L)}{\tau^{1/2}} + K'(r_L) = \frac{\hat{\hat{m}}}{r_L^2} y_{F_L} - \tau \frac{\hat{M}_L}{r_L^2}$$

It is concluded that the only term which can balance $\alpha H(r_L)/\tau^{1/2}$ is $\frac{\tau \hat{M}_L}{r_L^2}$ so that

$$H^{(0)}(r_L) = \tau^{3/2} \hat{M}_L^{(-1)} / r_L^2$$

where the assumed expansion for $\hat{M}_L$ is

$$\hat{M}_L = \sum_{1}^{-\infty} \alpha^n \hat{M}^{(-n)}$$

But $K^{(0)}(r_L) = 0$ so that

$$y_{F_L}^{(0)} = \tau^{3/2} M_L^{(-1)} / r_L^2 \qquad \text{(C-28)}$$

Consideration of the energy equation would show that

$$\hat{M}_L^{(-1)} = \varphi \left(\frac{\gamma - 1}{\gamma} \right) r_L^2 \tau^{-1/2} \qquad \text{(C-29)}$$

These conditions hold in the limit of $\omega \to \infty$ and the following conclusions may be drawn:

1. The surface mass flow perturbation goes to infinity as the square root of frequency and is larger for larger steady state mass flows.

2. The surface mass fraction perturbation remains finite thereby justifying partially the neglect of surface temperature perturbation.

It may be shown that a consistent procedure yields $G^{(0)} = L^{(0)} = 0$. Using the flame conditions in Eq. (C-19),

$$j \left[G'(\bar{r}_f) + \alpha G(\bar{r}_f) + K'(\bar{r}_f) \right] + L'(\bar{r}_f) + N'(\bar{r}_f) - \alpha L(\bar{r}_f) = 0$$

$$L(\bar{r}_f) + N(\bar{r}_f) = -\frac{\hat{\bar{m}}}{\bar{r}_f^2} j R_f = -j \left[G(\bar{r}_f) + K(\bar{r}_f) \right] \qquad \text{(C-30)}$$

Therefore, from Eq. (C-25)

$$R_f^{(0)} = \frac{-N^{(0)}(\bar{r}_f)\,\bar{r}_f^2}{\hat{\bar{m}}\;\dot{j}} = \frac{\hat{M}^{(-2)}}{\bar{r}_f^2} \tag{C-31}$$

Evaluating the perturbation form of Eq. (C-5) at the flame

$$M_F(\bar{r}_f) = y_F'(\bar{r}_f) + R_f\,Y''(\bar{r}_f)$$

or

$$M_F(\bar{r}_f) = G'(\bar{r}_f) + \alpha\,G(\bar{r}_f) + K'(\bar{r}_f) + \frac{\hat{\bar{m}}}{\bar{r}_f^3}\left(\frac{\hat{\bar{m}}}{\bar{r}_f} - 2\right)R_f$$

so that using Eq. (C-30)

$$M_F^{(0)'}(\bar{r}_f) = G^{(1)}(\bar{r}_f) + K^{(0)'}(\bar{r}_f) + \frac{\hat{\bar{m}}}{\bar{r}_f^5}\left(\frac{\hat{\bar{m}}}{\bar{r}_f} - 2\right)\hat{M}^{(-2)}(\bar{r}_f)$$

Using Eq. (C-30) to evaluate $G^{(1)}(\bar{r}_f)$,

$$M_F^{(0)} = K^{(1)}(\bar{r}_f) + \frac{1}{2}\left[\frac{N^{(0)'}(\bar{r}_f)}{\dot{j}} - K^{(0)'}(\bar{r}_f)\right] + \frac{\hat{M}^{(-2)}\hat{\bar{m}}}{\bar{r}_f^5}\left(\frac{\hat{\bar{m}}}{\bar{r}_f} - 2\right) \tag{C-32}$$

and the full mass burning rate perturbation

$$\dot{M}_F^{(0)}\left[r_f(t)\right] = M_F^{(0)}\,\bar{r}_f^2 + 2\,\hat{\bar{m}}\,\frac{R_f^{(0)}}{\bar{r}_f} \tag{C-33}$$

This solution depends upon the solution of the energy equation to determine $K^{(1)}(\bar{r}_f)$; however, for the purposes here it is not necessary to do so. Regardless of the magnitude of $\bar{r}_f$ (usually $\sim 10\,r_L$ for non-convective droplet burning) and regardless of $K^{(1)}(\bar{r}_f)$ Eq. (C-33) must be at least of order unity since $\hat{M}^{(-2)}(\bar{r}_f)$ is $O[\bar{r}_f^3]$. This shows that while the vaporization rate perturbation goes to infinity the burning rate remains finite of order unity, as the frequency tends to infinity and either is in or 180° out of phase with the pressure. This seeming paradox actually presents no problem since the <u>total</u> perturbation in mass released during any portion of a cycle is zero as frequency tends to infinity. In diffusion processes one cannot speak of rates at two different boundaries being equal since rates of storage in the interior would then be neglected.

APPENDIX D: WAVE SCATTERING FROM A SPHERE AT THE FORWARD STAGNATION POINT

In the inviscid field the equation for the perturbation velocity potential $\phi e^{i\omega t}$ defined by

$$\breve{V} = \nabla \phi e^{i\omega t}$$

is[1].

$$\nabla^2 \phi + (\omega M)^2 \phi = 0 \tag{D-1}$$

A boundary condition is that far from the body the velocity must be that of a plane wave given by Eq. (3.22)

$$\phi_{\breve{x}}(\infty) = e^{i\omega \delta M \breve{x}}$$

Since the normal velocity at the body must be zero, Neumann conditions hold.

$$\phi_n(\text{body}) = 0$$

A sphere will be assumed for the body so that the boundary conditions become

$$\phi_{\breve{x}}(\infty, \theta) = e^{i\omega M \delta r \left[\cos\left(\theta + \frac{\pi}{2}\right)\right]}$$

$$\phi_r(1, \theta) = 0 \tag{D-2}$$

where r is the radial variable in spherical coordinates, $\breve{x}$ is measured from the sphere center, and axial symmetry is

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1. Morse, P. M., and Feshbach, H., Methods of Theoretical Physics, Vol. I, McGraw Hill, New York, 1953, p. 163.

assumed. It is assumed $\omega M \ll 1$ such that a series expansion

$$\phi = \sum_{n=0}^{\infty} (i\omega\delta M)^n \phi^{(n)}$$

is attempted. Then Eq's. (D-1) and (D-2) become to first order in $i\omega\delta M$

$$\nabla^2 \phi^{(0)} = 0$$
$$\nabla^2 \phi^{(1)} = 0 \tag{D-3}$$

$$\phi^{(0)}_{\breve{x}}(\infty, \theta) = 1$$
$$\phi^{(0)}_{r}(1, \theta) = 0$$
$$\phi^{(1)}_{\breve{x}}(\infty, \theta) = \breve{x} = r\cos\left(\theta + \frac{\pi}{2}\right)$$
$$\phi^{(1)}_{r}(1, \theta) = 0 \tag{D-4}$$

It may be immediately verified that

$$\phi^{(0)} = \cos\left(\theta + \frac{\pi}{2}\right)\left(r + \frac{1}{2r^2}\right) \tag{D-5}$$

and

$$\phi^{(1)} = \frac{1}{6}\left(r^2 + \frac{2}{3r^2}\right)\left[3\cos^2\left(\theta + \frac{\pi}{2}\right) - 1\right] \tag{D-6}$$

are solutions of Eq's. (D-3) and (D-4).

The velocity at the surface is then given by

$$\breve{\vec{V}}(1, \theta) = \left\{ -\frac{3}{2}\sin\left(\theta + \frac{\pi}{2}\right) - i\omega\delta M \frac{5}{3}\cos\left(\theta + \frac{\pi}{2}\right)\sin\left(\theta + \frac{\pi}{2}\right) + O\left[(i\omega\delta M)^2\right] \right\} e^{i\omega t}$$

Then expanding for $\theta \sim \pi/2$ in the direction of increasing θ the boundary condition for the edge of the boundary layer in inner variables is

$$u(s,\infty) = \frac{3}{2} s \left[1 + \frac{10}{9} i\omega\delta M\right] e^{i\omega t} \qquad \text{(D-7)}$$

It should furthermore be clear that the first term of Eq. (D-7), representing the quasi-steady state, is also the solution for the steady state inviscid flow in the vicinity of the stagnation point. This is obvious since the zeroth order equations of Eq's. (D-3) and (D-4) are equations holding for the steady state velocity potential for $M^2 \ll 1$.

APPENDIX E: STAGNATION POINT HIGH FREQUENCY ANALYSIS

The equations

$$P''' + \frac{4}{3} F P'' - \frac{4}{3} F'P' + \frac{4}{3} F''P - \frac{4}{9} i\omega P' = \gamma\delta M F''' - \frac{4}{9}\Big[T(3+i\omega) + \frac{3}{2}\sigma + \frac{3}{2}\delta M T(\gamma-1) + \frac{10}{3} i\omega\,\delta M T\left(\frac{i\omega}{3}+1\right)\Big] \quad \text{(E-1a)}$$

$$\sigma'' + \frac{4}{3} F\sigma' - \frac{4}{9} i\omega\sigma = -\frac{4}{3} PT' + \frac{4}{9}\delta M\, i\omega(\gamma-1)T + \gamma\delta M T'' \quad \text{(E-1b)}$$

$$y_K'' + \frac{4}{3} F y_K' - \frac{4}{9} i\omega\, y_K = -\frac{4}{3} P Y_K' + \gamma\delta M Y_K'' \quad \text{(E-1c)}$$

are to be solved for high ω subject to the boundary conditions

<u>Surface</u>

$$P'(0) = \sigma(0) = 0$$

$$y_F(0) = y_{F_w}$$

$$y_F'(0) = \frac{4}{3}(1-Y_{F_w})[P(0) + \gamma\delta M F(0)] - \frac{4}{3} y_{F_w} F(0)$$

$$\sigma'(0) = -\frac{4}{3B_\infty}[P(0) + \gamma\delta M F(0)] \quad \text{(E-2)}$$

<u>Flame</u>

$$j\, y_F(\bar\eta_f) + y_o(\bar\eta_f) = 0$$

$$j\, y_F'(\bar\eta_f) + y_o'(\bar\eta_f) = 0$$

$$\sigma_F(\bar\eta_f) - \sigma_o(\bar\eta_f) + q\, y_F(\bar\eta_f) = 0$$

$$\sigma_F'(\bar\eta_f) - \sigma_o'(\bar\eta_f) + q\, y_F'(\bar\eta_f) = 0 \quad \text{(E-3)}$$

Infinity

$$P'(\infty) = 1$$
$$\sigma(\infty) = -\delta M(\gamma - 1)$$
$$y_c(\infty) = 0$$

(E-4)

Defining the high frequency parameter and variables

$$\alpha^2 = 1/i\omega \qquad \beta_1 = \frac{\zeta}{\alpha} \qquad \beta_2 = \frac{\zeta - \zeta_f}{\alpha} \qquad \text{(E-5)}$$

and assuming the existence of a uniformly valid composite expansion of the form

$$P = H(\zeta) + R_1(\beta_1) + R_2(\beta_2)$$
$$\sigma = S(\zeta) + U_1(\beta_1) + U_2(\beta_2)$$
$$y_K = V_K(\zeta) + W_{K_1}(\beta_1) + W_{K_2}(\beta_2)$$

(E-6)

Eq's. (E-1) become

$$H' - T = \frac{9}{4}\alpha^2\left\{H''' + \frac{4}{3}FH'' - \frac{4}{3}F'H' + \frac{4}{3}HF'' - \gamma\delta M F'''\right\}$$
$$+ \alpha^2\left\{T\left[3 + \frac{3}{2}\delta M(\gamma-1)\right] + \frac{3}{2}S\right\} + \frac{10}{3}\delta M T\left(\frac{1}{3\alpha^2} + 1\right) \qquad \text{(E-7a)}$$

$$S + \delta M(\gamma - 1)T = \frac{9}{4}\alpha^2\left\{S'' + \frac{4}{3}FS' + \frac{4}{3}HT' - \gamma\delta M T''\right\} \qquad \text{(E-7b)}$$

$$V = \frac{9}{4}\alpha^2\left\{V'' + \frac{4}{3}FV' + \frac{4}{3}HY_K' - \gamma\delta M Y_K''\right\} \qquad \text{(E-7c)}$$

$$R''' - \frac{4}{9}R' = -\frac{4}{3}\alpha F R'' + \frac{4}{3}\alpha F_\beta R' + \frac{4}{3}\alpha R F_\beta - \frac{2}{3}\alpha^3 U \qquad \text{(E-8a)}$$

$$U'' - \frac{4}{9}U = -\frac{4}{3}\alpha\left[FU' + RT_{\beta}\right]$$ (E-8b)

$$W'' - \frac{4}{9}W = -\frac{4}{3}\alpha\left[FW' + RY_{\kappa\beta}\right]$$ (E-8c)

Eq's. (E-8) are valid for either β_1 or β_2 and represent differential equations for the quantities important near boundaries. From their form it can be seen that solutions with exponential decay away from boundaries can be picked. Therefore, R_2 cannot influence R_1 to any finite power of α if η_f is $O[1]$.

Assuming the frequency expansion

$$R = \sum_{n=-2}^{\infty} \alpha^n r^{(n)}(\beta)$$
$$U = \sum_{n=0}^{\infty} \alpha^n u^{(n)}(\beta)$$
$$W_\kappa = \sum_{n=0}^{\infty} \alpha^n w_\kappa^{(n)}(\beta)$$
$$H = \sum_{n=-2}^{\infty} \alpha^n h^{(n)}(\eta)$$
$$S = \sum_{n=0}^{\infty} \alpha^n s^{(n)}(\eta)$$
$$V_\kappa = \sum_{n=0}^{\infty} \alpha^n v_\kappa^{(n)}(\eta)$$ (E-9)

Eq's. (E-7) and (E-8) become

$$h^{(-2)'} = \frac{10}{9}\delta MT$$
$$h^{(-1)'} = 0$$
$$h^{(0)'} = T\left[1 + \frac{10}{3}\delta M\right] + \frac{9}{4}\left[h^{(-2)'''} + \frac{4}{3}Fh^{(-2)''} - \frac{4}{3}F'h^{(-2)'} + \frac{4}{3}F''h^{(-2)}\right]$$
$$h^{(1)'} = 3F''h^{(-1)}$$
$$h^{(2)'} = \frac{9}{4}\left[h^{(0)'''} + \frac{4}{3}Fh^{(0)''} - \frac{4}{3}F'h^{(0)'} + \frac{4}{3}F''h^{(0)}\right] + \frac{3}{2}s^{(0)} - \frac{9}{4}\gamma\delta MF'''$$

$n \geq 3$

$$h^{(n)'} = \frac{9}{4}\left[h^{(n-2)'''} + \frac{4}{3}Fh^{(n-2)''} - \frac{4}{3}F'h^{(0)'} + \frac{4}{3}F''h^{(0)}\right] + \frac{3}{2}s^{(n-2)}$$ (E-10a)

$$s^{(0)} = -\delta M(\gamma-1)T + 3T' h^{(-2)}$$
$$s^{(1)} = 3h^{(-1)} T'$$
$$s^{(2)} = \tfrac{9}{4}\left\{ s^{(0)''} + \tfrac{4}{3} F s^{(0)'} + \tfrac{4}{3} h^{(0)} T' - \gamma\delta M T'' \right\}$$

$n \geq 3$

$$s^{(n)} = \tfrac{9}{4}\left\{ s^{(n-2)''} + \tfrac{4}{3} F s^{(n-2)'} + \tfrac{4}{3} h^{(n-2)} T' \right\} \qquad \text{(E-10b)}$$

$$v_K^{(0)} = 3h^{(-2)} Y_K' \qquad v_K^{(1)} = 3h^{(-1)} Y_K'$$
$$v_K^{(2)} = \tfrac{9}{4}\left\{ \tfrac{4}{3} h^{(0)} Y_K' - \gamma\delta M Y_K'' \right\}$$

$n \geq 3$

$$v_K^{(n)} = \tfrac{9}{4}\left\{ v_K^{(n-2)''} + \tfrac{4}{3} F v_K^{(n-2)'} + \tfrac{4}{3} h^{(n-2)} Y_K' \right\} \qquad \text{(E-10c)}$$

$$r^{(m)'''} - \tfrac{4}{9} r^{(m)'} = 0 \qquad \text{(E-11a)}$$

$$u^{(m)''} - \tfrac{4}{9} u^{(m)} = 0 \qquad \text{(E-11b)}$$

$$w_K^{(m)''} - \tfrac{4}{9} w_K^{(m)} = 0 \qquad \text{(E-11c)}$$

where m is the order of the first non-zero solutions.

If near the boundaries F has the expansions

$$F = \sum_{n=0}^{\infty} a_{w_n} (\alpha\beta_1)^n \quad ; \quad F = \sum_{n=0}^{\infty} a_{f_n} (\alpha\beta_2)^n$$

then higher order equations from Eq's (E-8) can be easily developed from Eq's. (E-11). From Eq's. (E-6) and (E-9) the boundary conditions, Eq's. (E-2)-(E-4), become

$$r_{F_i}^{(-2)'}(0) = 0$$

$n \geq -1$

$$h^{(n)'}(0) + r_{F_i}^{(n+1)'}(0) = 0 \qquad \text{(E-12a)}$$

$$s^{(n)}(0) + u_{F_i}^{(n)}(0) = 0 \qquad \text{(E-12b)}$$

$$v_F^{(n)}(0) + w_{F_i}^{(n)}(0) = y_{F_w}^{(n)} \qquad \text{(E-12c)}$$

$$h^{(-2)}(0) + r_{F_i}^{(-2)}(0) = 0$$

$$u_{F_i}^{(0)'} = -\frac{4}{3B_\infty}\left[h^{(-1)}(0) + r_{F_i}^{(-1)}(0)\right]$$

$$s^{(0)'}(0) + u_{F_i}^{(1)'}(0) = -\frac{4}{3B_\infty}\left[\gamma s M F(0) + h^{(0)}(0) + r_{F_i}^{(0)}(0)\right]$$

$n \geq 1$

$$s^{(n)'}(0) + u_{F_i}^{(n+1)'}(0) = -\frac{4}{3B_\infty}\left[h^{(n)}(0) + r_{F_i}^{(n)}(0)\right] \qquad \text{(E-12d)}$$

$$W_{F_i}^{(0)'} = \frac{4}{3}(1 - Y_{F_w})\left[h^{(-1)}(0) + r_{F_i}^{(-1)}(0)\right]$$

$$V_F^{(0)'}(0) + W_{F_i}^{(1)'}(0) = \frac{4}{3}(1 - Y_{F_w})\left[\gamma s M F(0) + h^{(0)}(0) + r_{F_i}^{(0)}(0)\right]$$

$n \geq 1$

$$V_F^{(n)'}(0) + W_{F_i}^{(n+1)'}(0) = \frac{4}{3}(1 - Y_{F_w})\left[h^{(n)}(0) + r_{F_i}^{(n)}(0)\right] \qquad \text{(E-12e)}$$

$$\dot{j}\left[v_F^{(n)} + w_{F_o}^{(n)}\right]\Big|_{\eta_f} + \left[v_o^{(n)} + w_{o_i}^{(n)}\right]\Big|_{\eta_f} = 0 \tag{E-13a}$$

$$\dot{j}\, w_{F_o}^{(0)'} + w_{o_i}^{(0)'} = 0$$

$n \geq 0$

$$\dot{j}\left[v_F^{(n)'} + w_{F_o}^{(n+1)'}\right]\Big|_{\eta_f} + \left[v_o^{(n)'} + w_{o_i}^{(n+1)'}\right] = 0 \tag{E-13b}$$

$$\left[s_F^{(n)} + u_{F_o}^{(n)} - s_o^{(n)} - u_{o_i}^{(n)}\right]\Big|_{\eta_f} + q\left[v_F^{(n)} + w_{F_o}^{(n)}\right]\Big|_{\eta_f} = 0 \tag{E-13c}$$

$$\left(u_{F_o}^{(0)'} - u_{o_i}^{(0)'} + q\, w_{F_o}^{(0)'}\right)\Big|_{\eta_f} = 0$$

$n \geq 0$

$$\left[s_F^{(n)'} + u_{F_o}^{(n+1)'} - s_o^{(n)'} - u_{o_i}^{(n+1)'}\right]\Big|_{\eta_f} + q\left[v_F^{(n)'} + w_{F_o}^{(n+1)'}\right]\Big|_{\eta_f} = 0 \tag{E-13d}$$

$$\left[h_F^{(n)} + r_{F_o}^{(n)}\right]\Big|_{\eta_f} = \left[h_o^{(n)} + r_{o_i}^{(n)}\right]\Big|_{\eta_f} \tag{E-13e}$$

$$r_{F_o}^{(0)'}(\eta_f) = r_{o_i}^{(0)'}(\eta_f)$$

$$h_F^{(n)'}(\eta_f) + r_{F_o}^{(n+1)'}(\eta_f) = h_o^{(n)'}(\eta_f) + r_{o_i}^{(n+1)'}(\eta_f) \tag{E-13f}$$

$$r_{F_o}^{(0)''}(\eta_f) = r_{o_i}^{(0)''}(\eta_f) \qquad r_{F_o}^{(1)''}(\eta_f) = r_{o_i}^{(1)''}(\eta_f)$$

$$h_F^{(n)''}(\eta_f) + r_{F_o}^{(n+2)''}(\eta_f) = h_o^{(n)''}(\eta_f) + r_{o_i}^{(n+2)''}(\eta_f) \tag{E-13g}$$

$$h^{(-2)'}(\infty) = \delta M \tfrac{10}{9} \qquad h^{(-1)'}(\infty) = 0 \qquad h^{(0)'}(\infty) = 1 \qquad n \geq 1\ \ h^{(n)'}(\infty) = 0 \tag{E-14a}$$

$$\sigma^{(0)}(\infty) = -\delta M(\gamma - 1) \qquad n \geq 1 \quad \sigma^{(n)}(\infty) = 0 \tag{E-14b}$$

$$y_o^{(n)}(\infty) = 0 \tag{E-14c}$$

These conditions are all valid to $O[\epsilon^{-1/6}]$. Consider first the problem at the wall. Eq's. (E-10a) yield

$$h^{(-2)} = \frac{10}{9} \delta M \int_0^{\eta} T \, d\eta + A_{h^{(-2)}}$$

$$h^{(-1)} = A_{h^{(-1)}}$$

Choosing $r_{F_i}^{(-2)} = 0$, Eq. (E-11a) yields

$$r_{F_i}^{(-1)} = + A_{r_{F_i}^{(-1)}} e^{-2/3 \beta_1}$$

Application of the no slip condition, Eq. (E-12a), yields

$$A_{r_{F_i}^{(-1)}} = -\frac{10}{9} \delta M \tau$$

By Eq's. (E-10b)

$$s^{(0)} = -\delta M(\gamma - 1) T + 3 T' h^{(-2)}$$

Then Eq's. (E-12b) say

$$A_{h^{(-2)}} = 0$$

From Eq. (E-11b)

$$u_{F_i}^{(0)} = A_{u_{F_i}^{(0)}} e^{-2/3 \beta_1}$$

Application of Eq. (E-12b) yields

$$A_{u_{F_i}^{(0)}} = \delta M(\gamma - 1) \tau$$

Then Eq's. (E-12d) yield $A_{h^{(-1)}}$. Continuing, the procedure is as follows:

1. Solve all of Eq's. (E-10) in terms of the constants of integration, $A_{h^{(n)}}$.
2. Solve the higher order counterparts of Eq's. (E-11) choosing exponential decay into the region.
3. Eq's. (E-12a) determine $A_{r_{F_i}^{(n)}}$.
4. Eq. (E-12b) determine $A_{u_{F_i}^{(n+1)}}$.
5. Eq's. (E-12d) determine $A_{h^{(n)}}$.
6. Eq's. (E-12e) determine $A_{w_{F_i}^{(n+1)}}$.

Now consider the flame problem. Eq's.(E-13) state the r solutions are zero up to and including α^{-1}. Eq's. (E-13f and g) determine the r's for α^0 and Eq's. (E-13e) determine the jump in $A_{h^{(0)}}$. This procedure is repeated to determine all $A_{h^{(n)}}$, $A_{r_{F_0}^{(n)}}$, and $A_{r_{O_i}^{(n)}}$. Now Eq's. (E-13a and b) and (E-10c) say that the w's are zero up to α. Similarly the u's are zero up to α. Applying these conditions and using Eq. (E-11c),

$$W_{F_0}^{(2)} = A_{W_{F_0}^{(2)}} e^{2/3\ \beta_2} \qquad W_{O_i}^{(2)} = A_{W_{O_i}^{(2)}} e^{-\frac{2}{3}\beta_2}$$

and

$$A_{W_{O_i}^{(2)}} = \dot{j} A_{W_{F_0}^{(2)}} = \frac{15}{2}\alpha M q (Y_F')^2$$

Similarly, the recursive procedure to determine the complete flame problem is as follows:

1. Since all $A_{r_{F_0}^{(n)}}$, $A_{r_{O_i}^{(n)}}$, and $A_{h^{(n)}}$ are are known all $V^{(n)}$ and $\mathcal{B}^{(n)}$ are known.
2. Eq's. (E-13c and d) determine $A_{u_{F_0}^{(n)}}$ and $A_{u_{O_i}^{(n)}}$.
3. Eq's. (E-13a and b) determine $A_{W_{F_0}^{(n)}}$ and $A_{W_{O_i}^{(n)}}$.

Solutions to Eq's (E-11) are always picked to have exponential decay into the region of interest. Eq's. (E-14a) are automatically satisfied by Eq's. (E-10).

Following this procedure, the mass burning rate perturbation at the flame is evaluated. Tracing through the stagnation point transformations Eq. (3.12c) becomes

$$m_{F_f} = -\frac{3}{2}\gamma M^2 p\left[\bar{Y}_F' + \epsilon\left(\tilde{\eta}_f \bar{Y}_F'' + y_F'\right)e^{i\omega t}\right]_f$$

Therefore,

$$\frac{M_{F_f}}{\bar{m}_{F_f}} = \delta M\left(\frac{10}{3}T_f - \gamma\right) - \alpha\,\delta M\,5\,\bar{m}_{F_f}q + \alpha^2 3T_f + O[\alpha^2\delta M] + O[\alpha^3]$$

(E-15)

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